This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

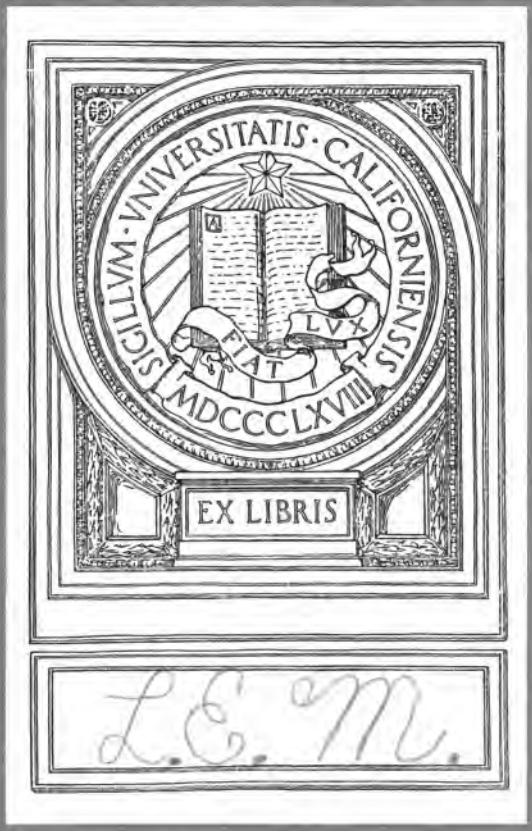
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

UC-NRLF
$B 278 189

Digitized by Google



Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

THE

ESSENTIALS

OF

PLANE AND SPHERICAL TRIGONOMETRY.

BY

WEBSTER WELLS, S.B.,

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

DISCARDED BY THE UNIVERSITY OF CALIFORNIA

LEACH, SHEWELL, AND SANBORN.

BOSTON. NEW YORK. CHICAGO.

Digitized by Google

COPYRIGHT, 1887,
BY WEBSTER WELLS.

TYPOGRAPHY BY
S. CUSHING & CO.

PRESSWORK BY
BERWICK & SMITH.

Digitized by Google

PREFACE.

THIS volume contains only those portions of Plane and Spherical Trigonometry which are essential in the practical applications of the subject to problems in surveying, geodesy, and navigation.

The attention of teachers and others is invited to the following, which may be regarded as constituting the salient features of the work:

1. The proofs of the functions of 0°, 90°, 180°, and 270°; Arts. 38 to 41.

2. The figures, and the tabular arrangement of the work, in the discussions of Arts. 42 and 44.

3. The solution of problems in Art. 56 by the construction of a diagram, in a manner analogous to that of Art. 15.

4. The proofs of the fundamental formulæ for any angle; Arts. 57 to 59.

5. The discussion of the line values of the trigonometric functions, and their application in tracing the changes in the functions as the angle increases from 0° to 360°; Arts. 62 to 64.

6. The proofs of the formulæ for the sines and cosines of $x + y$ and $x - y$ for any values of x and y; Arts. 68 to 70.

7. The discussion of the *ambiguous case* in the solution of plane oblique triangles; Arts. 124 to 126.

184018

Digitized by Google

8. The *geometrical* proofs (Art. 138) of the propositions that in any spherical right triangle:

I. If the sides including the right angle are in the same quadrant, the hypotenuse is $< 90°$; if they are in different quadrants, the hypotenuse is $> 90°$.

II. An angle is in the same quadrant as its opposite side.

9. The discussion of the properties of spherical right triangles *before* those of spherical oblique triangles; see Chapters XI. and XII.

10. The reduction of the number of cases in the complete demonstration of the fundamental formulæ for spherical right triangles, to three, by application of the results proved geometrically in Art. 138; see Art. 143.

11. The discussions of the *ambiguous cases* in the solution of spherical oblique triangles (Arts. 171 and 172); especially the rules given on pages 130 and 132 for determining the number of solutions.

At the end of the book will be found a collection of formulæ in form for convenient reference.

Teachers who desire a briefer course are recommended to omit Chapter IV., which may be done without interrupting the logical completeness of the rest of the work. Chapter VI. may also be omitted by those who have taken the subject of Logarithms in their course in Algebra. The course might be still further abridged, if desired, by the omission of the exercises at the end of Chapter V.

WEBSTER WELLS.

BOSTON, 1887.

Digitized by Google

CONTENTS.

PART I. PLANE TRIGONOMETRY.

		Page
I.	Definitions; Measurement of Angles	1
	Circular Measure of an Angle	1
II.	The Trigonometric Functions	4
	Functions of Acute Angles	4
	Fundamental Theorems	10
III.	Application of Algebraic Signs	12
	Trigonometric Functions of Angles in General	12
	Rectangular Co-ordinates	14
	General Definitions of the Functions	15
	Functions of 0°, 90°, 180°, 270°, and 360°	19
	Functions of $(-A)$ in Terms of those of A	21
	Functions of $(90° + A)$ in Terms of those of A	24
	Proofs of the Fundamental Formulæ for any Angle	34
IV.	Miscellaneous Theorems	36
	To express Each of the Six Principal Functions in Terms of the other Five	36
	Limiting Values of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$	37
	Line Values of the Trigonometric Functions	38
V.	General Formulæ	42
	Functions of $2x$	48
	Functions of $\frac{1}{2}x$	49
	Inverse Trigonometric Functions	50
VI.	Logarithms	54
	Properties of Logarithms	56
	Applications	61
	Arithmetical Complement	63
VII.	Solution of Right Triangles	67
	Formulæ for the Area of a Right Triangle	73

Digitized by Google

PAGE

VIII. GENERAL PROPERTIES OF TRIANGLES 75
Formulæ for the Area of an Oblique Triangle . . 81

IX. SOLUTION OF OBLIQUE TRIANGLES 83
Area of an Oblique Triangle 92

PART II. SPHERICAL TRIGONOMETRY.

X. GEOMETRICAL DEFINITIONS AND PRINCIPLES . . . 95

XI. SPHERICAL RIGHT TRIANGLES 99
Napier's Rules of Circular Parts 104
Solution of Spherical Right Triangles 106
Solution of Quadrantal Triangles 112

XII. SPHERICAL OBLIQUE TRIANGLES 114
General Properties of Spherical Triangles 114
Napier's Analogies 121
Solution of Spherical Oblique Triangles 123
Applications 133

FORMULÆ. 137
ANSWERS TO THE EXAMPLES 144

Digitized by Google

LIBRARY OF THE UNIVERSITY OF CALIFORNIA

PART I.

PLANE TRIGONOMETRY.

I. DEFINITIONS; MEASUREMENT OF ANGLES.

1. *Trigonometry* is that branch of mathematics in which algebraic processes are used to treat of the properties and measurement of geometrical figures.

In *Plane Trigonometry* we consider *plane* figures only.

2. An angle is measured by finding its ratio to another angle adopted arbitrarily as the unit of measurement.

3. The usual unit of measurement for angles is the *degree*, or an angle equal to the ninetieth part of a right angle.

To express fractional parts of the unit, the degree is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

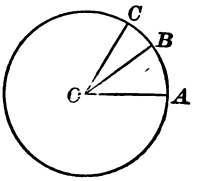
Degrees, minutes, and seconds are denoted by the symbols °, ′, ″, respectively; thus, 43° 22′ 37″ denotes an angle of 43 degrees, 22 minutes, and 37 seconds.

CIRCULAR MEASURE OF AN ANGLE.

4. Another method of measuring angles, and one of great importance, is known as the *circular method*, in which the unit of measurement is *the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.*

Digitized by Google

5. Let AOB be any angle, and AOC the unit of circular measure.



By Geometry, we have

$$\frac{\text{angle } AOB}{\text{angle } AOC} = \frac{\text{arc } AB}{\text{arc } AC}.$$

Whence, circular measure $AOB = \dfrac{\text{arc } AB}{OA}$ (Art. 4).

That is, *the circular measure of an angle is equal to the ratio of its subtending arc to the radius of the circle.*

For example, the circular measure of a right angle is equal to the ratio of one-fourth the circumference to the radius.

But the circumference of a circle is equal to the radius multiplied by 2π, where $\pi = 3.14159265\ldots$

Hence, if R denotes the radius, we have

$$\text{circular measure of } 90° = \frac{\frac{1}{4} \text{ of } 2\pi R}{R} = \frac{\pi}{2}.$$

6. Since the circular measure of 90° is $\frac{\pi}{2}$, the circular measure of 180° is π; of 60°, $\frac{\pi}{3}$; of 45°, $\frac{\pi}{4}$; of 30°, $\frac{\pi}{6}$; etc.

That is, *an angle expressed in degrees may be reduced to circular measure by finding its ratio to* 180°, *and multiplying the result by* π.

Thus, since 115° is $\frac{23}{36}$ of 180°, the circular measure of 115° is $\frac{23\pi}{36}$.

Digitized by Google

7. Conversely, *any angle expressed in circular measure may be reduced to degrees by multiplying by* 180° *and dividing by* π; or, more briefly, *by substituting* 180° *for* π.

Thus, $\frac{7\pi}{15} = \frac{7}{15}$ of $180° = 84°$.

8. In the circular method such expressions may occur as "the angle $\frac{2}{3}$," "the angle 1," etc.

These refer to the unit of circular measure; that is, the angle $\frac{2}{3}$ means an angle whose subtending arc is two-thirds of the radius.

The angle 1, that is, the angle whose subtending arc is equal to the radius, or the unit of circular measure, if reduced to degrees by the rule of Art. 7, gives

$$\frac{180°}{\pi} = \frac{180°}{3.14159265\ldots} = 57.2957795°\ldots$$

The rule of Art. 7 may then be modified as follows:

Any angle expressed in circular measure may be reduced to degrees by multiplying by 57.2957795°...

Thus, the angle $\frac{2}{3} = \frac{2}{3} \times 57.2957795°\ldots$

$= 38.1971863°\ldots = 38° 11' 49.87068''\ldots$

EXAMPLES.

9. Express the following in circular measure:

1. 135°. **3.** 11° 15′. **5.** 29° 15′. **7.** 128° 34$\frac{2}{7}$′.

2. 198°. **4.** 37° 30′. **6.** 174° 22′ 30″. **8.** 92° 48′ 45″.

Express the following in degree measure:

9. $\frac{1}{2}$. **11.** $\frac{37\pi}{30}$. **13.** $\frac{3}{4}$. **15.** $\frac{2\pi - 1}{3}$.

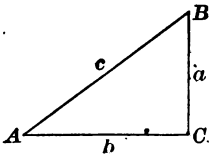
10. $\frac{3\pi}{5}$. **12.** $\frac{5\pi}{4}$. **14.** 2. **16.** $\frac{\pi - 1}{4}$.

Digitized by Google

II. THE TRIGONOMETRIC FUNCTIONS.

FUNCTIONS OF ACUTE ANGLES.

10. Let BAC be any acute angle.



From any point in either side, as B, draw BC perpendicular to the other side, forming a right triangle ABC.

We then have the following definitions, applicable to either of the acute angles A and B:

In any right triangle,

The SINE *of either acute angle is the ratio of the opposite side to the hypotenuse.*

The COSINE *is the ratio of the adjacent side to the hypotenuse.*

The TANGENT *is the ratio of the opposite side to the adjacent side.*

The COTANGENT *is the ratio of the adjacent side to the opposite side.*

The SECANT *is the ratio of the hypotenuse to the adjacent side.*

The COSECANT *is the ratio of the hypotenuse to the opposite side.*

That is, denoting the sides BC, CA, and AB by a, b, and c, and employing the usual abbreviations,

$$\left.\begin{array}{lll} \sin A = \frac{a}{c}, & \tan A = \frac{a}{b}, & \sec A = \frac{c}{b}, \\ \cos A = \frac{b}{c}, & \cot A = \frac{b}{a}, & \csc A = \frac{c}{a}. \end{array}\right\} \quad (1)$$

Digitized by Google

And in like manner,

$$\left.\begin{array}{lll} \sin B = \frac{b}{c}, & \tan B = \frac{b}{a}, & \sec B = \frac{c}{a}, \\ \cos B = \frac{a}{c}, & \cot B = \frac{a}{b}, & \operatorname{csc} B = \frac{c}{b}. \end{array}\right\} \quad (2)$$

11. The following definitions are also used:

The versed sine of an angle is equal to unity minus the cosine of the angle.

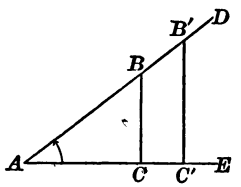
The coversed sine is equal to unity minus the sine.

That is, $\operatorname{vers} A = 1 - \frac{b}{c}$, $\operatorname{covers} A = 1 - \frac{a}{c}$,

$\operatorname{vers} B = 1 - \frac{a}{c}$, $\operatorname{covers} B = 1 - \frac{b}{c}$.

12. The eight ratios defined in Arts. 10 and 11 are called *Trigonometric Functions*, or *Trigonometric Ratios*, of the angle.

It is important to observe that their values depend solely on the magnitude of the angle, and are entirely independent of the lengths of the sides of the right triangle which contains it.



Thus, let B and B' be any two points in the side AD of the angle DAE, and draw BC and $B'C'$ perpendicular to AE.

Then by Art. 10, we have

$$\sin A = \frac{BC}{AB}, \text{ and } \sin A = \frac{B'C'}{AB'}.$$

Digitized by Google

But the right triangles ABC and $AB'C'$ are similar, since they have the angle A common; and therefore, by Geometry,

$$\frac{BC}{AB} = \frac{B'C'}{AB'}.$$

Thus the two values obtained for sin A are seen to be equal.

13. We obtain from (1) and (2), Art. 10,

$$a = c \sin A, \qquad b = c \sin B,$$
$$b = c \cos A, \qquad a = c \cos B.$$

That is, *in any right triangle, either side about the right angle is equal to the hypotenuse multiplied by the sine of the opposite angle, or by the cosine of the adjacent angle.*

14. We have from Arts. 10 and 11,

$$\sin A = \frac{a}{c} = \cos B, \qquad \sin B = \frac{b}{c} = \cos A,$$

$$\tan A = \frac{a}{b} = \cot B, \qquad \tan B = \frac{b}{a} = \cot A,$$

$$\sec A = \frac{c}{b} = \csc B, \qquad \sec B = \frac{c}{a} = \csc A,$$

$$\text{vers } A = 1 - \frac{b}{c} = \text{covers } B, \qquad \text{vers } B = 1 - \frac{a}{c} = \text{covers } A.$$

Since the angles A and B are complements of each other, the above results may be expressed as follows:

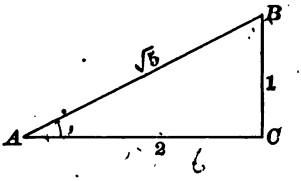
The sine, tangent, secant, and versed sine of an acute angle are respectively the cosine, cotangent, cosecant, and coversed sine of the complement of the angle.

It is from this circumstance that the names *co*-sine, *co*-tangent, etc., were derived.

15. The Pythagorean Theorem affords a simple method for finding the values of the remaining seven functions of an acute angle, when the value of any one is given.

Digitized by Google

1. Given $\cot A = 2$; required the values of the remaining functions of A.



The equation may be written $\cot A = \frac{2}{1}$.

Then since the cotangent is the ratio of the adjacent side to the opposite side, we may regard our value as having been taken from a right triangle ABC, having its side AC adjacent to the angle A equal to 2 units, and its side BC opposite to A equal to 1 unit.

But by Geometry, we have

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{4+1} = \sqrt{5}.$$

Whence by definition,

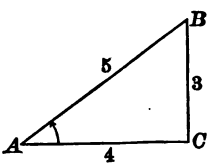
$$\sin A = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \qquad \sec A = \frac{\sqrt{5}}{2},$$

$$\cos A = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}, \qquad \csc A = \sqrt{5},$$

$$\tan A = \frac{1}{2}, \qquad \operatorname{vers} A = 1 - \frac{2\sqrt{5}}{5},$$

$$\operatorname{covers} A = 1 - \frac{\sqrt{5}}{5}.$$

2. Given $\operatorname{covers} A = \frac{2}{5}$; required the values of the remaining functions of A.



Digitized by Google

Since covers $A = 1 - \sin A$, we have $\sin A = 1 - \frac{2}{5} = \frac{3}{5}$.

We then take the opposite side BC equal to 3 units, and the hypotenuse AB equal to 5 units.

Then, $AC = \sqrt{\overline{AB}^2 - \overline{BC}^2} = \sqrt{25 - 9} = 4$.

Whence by definition,

$$\cos A = \frac{4}{5}, \qquad \cot A = \frac{4}{3}, \qquad \csc A = \frac{5}{3},$$

$$\tan A = \frac{3}{4}, \qquad \sec A = \frac{5}{4}, \qquad \text{vers } A = \frac{1}{5}.$$

EXAMPLES.

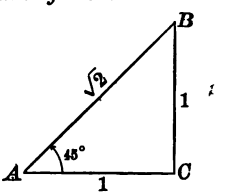
In each case find the values of the remaining functions:

3. $\tan A = \frac{2}{3}$. **6.** $\text{vers } A = \frac{1}{4}$. **9.** $\cot A = x$.

4. $\text{covers } A = \frac{3}{5}$. **7.** $\sin A = \frac{x}{y}$. **10.** $\cos A = \frac{8}{17}$.

5. $\csc A = 4$. **8.** $\sec A = \frac{13}{5}$. **11.** $\sec A = \frac{\sqrt{a^2 + b^2}}{b}$.

16. *To find the values of the sine, cosine, tangent, cotangent, secant, and cosecant of* 45°.



Let ABC be an isosceles right triangle, having each of the sides AC and BC equal to 1.

Then, $AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{1 + 1} = \sqrt{2}$.

Also, the angles A and B are equal; and since their sum is a right angle, each is equal to 45°.

Digitized by Google

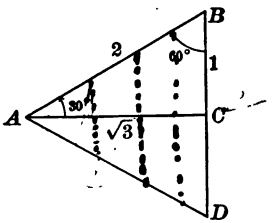
Then by definition,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \qquad \tan 45° = 1. \qquad \sec 45° = \sqrt{2}.$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \qquad \cot 45° = 1. \qquad \csc 45° = \sqrt{2}.$$

The second line might have been derived from the first by aid of Art. 14, since 45° is complement of itself.

17. *To find the values of the sine, cosine, tangent, cotangent, secant, and cosecant of* 30° *and* 60°.



Let ABD be an equilateral triangle, having each side equal to 2. Draw AC perpendicular to BD; then by Geometry,

$$BC = \tfrac{1}{2}BD = 1, \text{ and } \angle BAC = \tfrac{1}{2}\angle BAD = 30°.$$

Again, $AC = \sqrt{\overline{AB}^2 - \overline{BC}^2} = \sqrt{4-1} = \sqrt{3}$.

Then from the triangle ABC, by definition,

$$\sin 30° = \frac{1}{2} = \cos 60°. \qquad \cos 30° = \frac{\sqrt{3}}{2} = \sin 60°.$$

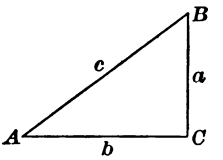
$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \cot 60°. \qquad \cot 30° = \sqrt{3} = \tan 60°.$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \csc 60°. \qquad \csc 30° = 2 = \sec 60°.$$

Or the functions of 60° may be derived from those of 30° by aid of Art. 14, since 60° is the complement of 30°.

Digitized by Google

FUNDAMENTAL THEOREMS.



18. We obtain from the definitions of Art. 10,

$$\sin A \csc A = \frac{a}{c} \times \frac{c}{a} = 1.$$

$$\cos A \sec A = \frac{b}{c} \times \frac{c}{b} = 1.$$

$$\tan A \cot A = \frac{a}{b} \times \frac{b}{a} = 1.$$

These results may be written

$$\left.\begin{aligned} \sin A &= \frac{1}{\csc A}, \quad \tan A = \frac{1}{\cot A}, \quad \sec A = \frac{1}{\cos A}, \\ \cos A &= \frac{1}{\sec A}, \quad \cot A = \frac{1}{\tan A}, \quad \csc A = \frac{1}{\sin A}. \end{aligned}\right\} \quad (3)$$

That is, *the sine of an acute angle is the reciprocal of the cosecant, the tangent is the reciprocal of the cotangent, and the secant is the reciprocal of the cosine.*

19. *To prove the formula*

$$\sin^2 A + \cos^2 A = 1.$$

Note. $\mathrm{Sin}^2 A$ signifies $(\sin A)^2$; that is, the square of the sine of A

By Geometry, $$a^2 + b^2 = c^2.$$

Dividing by c^2, $$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Digitized by Google

Whence by definition,

$$(\sin A)^2 + (\cos A)^2 = 1;$$

or,
$$\sin^2 A + \cos^2 A = 1. \qquad (4)$$

The result may be written in the forms

$$\sin A = \sqrt{1 - \cos^2 A}, \text{ and } \cos A = \sqrt{1 - \sin^2 A}.$$

20. *To prove the formulæ*

$$\tan A = \frac{\sin A}{\cos A}, \textit{ and } \cot A = \frac{\cos A}{\sin A}.$$

By definition,
$$\tan A = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A}, \qquad (5)$$

and
$$\cot A = \frac{b}{a} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\cos A}{\sin A}. \qquad (6)$$

21. *To prove the formulæ*

$$\sec^2 A = 1 + \tan^2 A, \textit{ and } \csc^2 A = 1 + \cot^2 A.$$

By Geometry,
$$c^2 = a^2 + b^2.$$

Dividing by b^2,
$$\frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}.$$

That is,
$$\sec^2 A = 1 + \tan^2 A. \qquad (7)$$

Dividing by a^2,
$$\frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}.$$

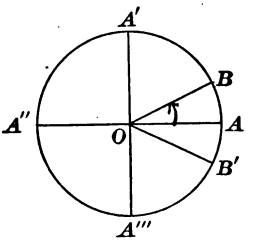
That is,
$$\csc^2 A = 1 + \cot^2 A. \qquad (8)$$

Digitized by Google

III. APPLICATION OF ALGEBRAIC SIGNS.

TRIGONOMETRIC FUNCTIONS OF ANGLES IN GENERAL.

22. In Geometry we are, as a rule, concerned with angles less than two right angles; but in Trigonometry it is convenient to regard them as unrestricted in magnitude.



In the circle AA'', let AA'' and $A'A'''$ be a pair of perpendicular diameters.

Suppose a radius OB to start from the position OA, and revolve about the point O as a pivot in the direction of OA'.

When it coincides with OA', it has described an angular magnitude of 90°; when it coincides with OA'', of 180°; with OA''', of 270°; with OA, its starting-point, of 360°; with OA' again, of 450°; etc. We thus see that a significance may be attached to any positive angular magnitude.

23. The interpretation of an angle as the measure of the amount of rotation of a moving radius, enables us to distinguish between positive and negative angles.

Thus, if a positive angle is taken to indicate revolution from the position OA in the direction of OA', a negative angle would naturally be taken as signifying revolution from the position OA' in the *opposite* direction, or towards OA'''.

Digitized by Google

That is, if the angles AOB and AOB' are each equal to 30° in absolute value, we should say that $AOB = +30°$, and $AOB' = -30°$.

We may thus conceive of *negative* angles of any magnitude whatever. It is immaterial which direction we consider the positive direction of rotation; but having adopted a certain direction as positive, our subsequent operations must be in accordance.

24. The fixed line OA, from which the rotation is supposed to commence, is called the *initial line*, and the final position of the rotating radius is called the *terminal line*.

Either side of an angle may be taken as the initial line, the other being then the terminal line; thus, in the angle AOB, we may consider OA the initial line and OB the terminal line, in which case the angle is positive; or we may consider OB the initial line and OA the terminal line, in which case the angle is negative.

25. In designating an angle, we shall always write first the letter at the extremity of the initial line; thus, in designating the angle AOB, if we regard OA as the initial line, we should call it AOB, and if we regard OB as the initial line, we should call it BOA.

26. There are always two angles in absolute value less than 360°, one positive and the other negative, formed by a given initial and terminal line.

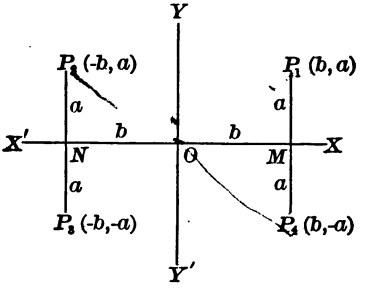
Thus, there are formed by OA and OB' the positive angle AOB' greater than 270°, and the negative angle AOB' less than 90°.

We shall distinguish between such angles by referring to them as "the positive angle AOB'," and "the negative angle AOB'," respectively.

27. It is evident that the terminal lines of two angles which differ by a multiple of 360° are coincident; thus, the angles 30°, 390°, −330°, etc., have the same terminal line.

Digitized by Google

RECTANGULAR CO-ORDINATES.



28. Let XX' and YY' be a pair of straight lines at right angles to each other; let P_1 be any point in their plane, and draw P_1M perpendicular to XX'.

The distances OM and P_1M are called the *rectangular co-ordinates* of P_1 with reference to the lines XX' and YY'.

OM is called the *abscissa*, and P_1M the *ordinate;* the lines XX' and YY' are called the *axes of X and Y*, respectively, and their intersection O is called the *origin*.

29. If in the above figure, $OM = ON = b$, and the perpendiculars $P_1M = P_2N = P_3N = P_4M = a$, the points P_1, P_2, P_3, and P_4 will have the same co-ordinates.

To avoid this ambiguity, the following conventions have been adopted:

Abscissas measured to the *right* of O are considered *positive*, and to the *left*, *negative*.

Ordinates measured *above* the line XX' are considered *positive*, and *below*, *negative*.

Then the co-ordinates of the four points will be:

Point.	*Abscissa.*	*Ordinate.*
P_1	b	a
P_2	$-b$	a
P_3	$-b$	$-a$
P_4	b	$-a$

Digitized by Google

Note. In the figures of this chapter, the small letters denote the *lengths* of the lines, without regard to their algebraic sign.

30. It is customary to denote the abscissa and ordinate of a point by the letters x and y, respectively. Thus the fact that the abscissa of a point is equal to b and its ordinate to a, is expressed by the saying that for the point in question $x = b$ and $y = a$.

The same fact may be stated more concisely by referring to the point as "the point (b, a)", where the first quantity in the parenthesis is understood to be the abscissa, and the second the ordinate.

31. If a point lies upon the axis of X, its ordinate is zero; and the same is true of the abscissa of a point upon the axis of Y.

GENERAL DEFINITIONS OF THE FUNCTIONS.

32. We will now give general definitions for the trigonometric functions, applicable to any angle whatever.

Take the initial line as the positive direction of the axis of X, the vertex being the origin.

Take any point in the terminal line, and construct its rectangular co-ordinates by dropping a perpendicular to the initial line, produced if necessary.

Then, designating the distance of the assumed point from the origin as the "distance" of the point,

The SINE	*is the ratio of the* ORDINATE *to the* DISTANCE.
The COSINE	*is the ratio of the* ABSCISSA *to the* DISTANCE.
The TANGENT	*is the ratio of the* ORDINATE *to the* ABSCISSA.
The COTANGENT	*is the ratio of the* ABSCISSA *to the* ORDINATE.
The SECANT	*is the ratio of the* DISTANCE *to the* ABSCISSA.
The COSECANT	*is the ratio of the* DISTANCE *to the* ORDINATE.

Note. These definitions include those of Art. 10. The definitions of the versed sine and coversed sine, given in Art. 11, are sufficiently general to apply to any angle whatever.

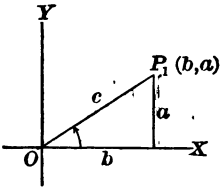
Digitized by Google

33. We will now apply the definitions of Art. 32 to the angles XOP_1, XOP_2, XOP_3, and XOP_4 in the following figures:

Let P_1, P_2, P_3, and P_4 be any points on the terminal lines OP_1, OP_2, OP_3, and OP_4, and let their co-ordinates be (b, a), $(-b, a)$, $(-b, -a)$, and $(b, -a)$, respectively.

Let $OP_1 = OP_2 = OP_3 = OP_4 = c$.

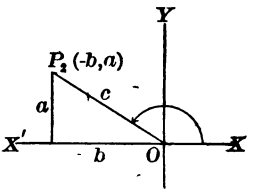
Then by the definitions,



$$\sin XOP_1 = \frac{a}{c}. \qquad \cos XOP_1 = \frac{b}{c}.$$

$$\tan XOP_1 = \frac{a}{b}. \qquad \cot XOP_1 = \frac{b}{a}.$$

$$\sec XOP_1 = \frac{c}{b}. \qquad \csc XOP_1 = \frac{c}{a}.$$

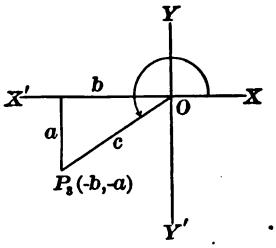


$$\sin XOP_2 = \frac{a}{c}. \qquad \cos XOP_2 = \frac{-b}{c} = -\frac{b}{c}.$$

$$\tan XOP_2 = \frac{a}{-b} = -\frac{a}{b}. \qquad \cot XOP_2 = \frac{-b}{a} = -\frac{b}{a}.$$

$$\sec XOP_2 = \frac{c}{-b} = -\frac{c}{b}. \qquad \csc XOP_2 = \frac{c}{a}.$$

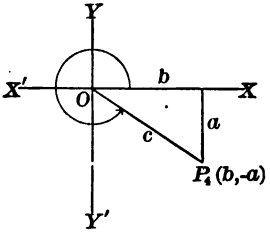
Digitized by Google



$\sin XOP_3 = \frac{-a}{c} = -\frac{a}{c}$; $\cos XOP_3 = \frac{-b}{c} = -\frac{b}{c}$.

$\tan XOP_3 = \frac{-a}{-b} = \frac{a}{b}$. $\cot XOP_3 = \frac{-b}{-a} = \frac{b}{a}$.

$\sec XOP_3 = \frac{c}{-b} = -\frac{c}{b}$. $\csc XOP_3 = \frac{c}{-a} = -\frac{c}{a}$.



$\sin XOP_4 = \frac{-a}{c} = -\frac{a}{c}$. $\cos XOP_4 = \frac{b}{c}$.

$\tan XOP_4 = \frac{-a}{b} = -\frac{a}{b}$. $\cot XOP_4 = \frac{b}{-a} = -\frac{b}{a}$.

$\sec XOP_4 = \frac{c}{b}$. $\csc XOP_4 = \frac{c}{-a} = -\frac{c}{a}$.

34. Since the terminal lines of two angles which differ by a multiple of 360° are coincident (Art. 27), it is evident that the trigonometric functions of two such angles are identical.

Thus, the functions of 50°, 410°, 770°, −310°, etc., are identical.

It is customary to express this fact by saying that the trigonometric functions are *periodic*.

Digitized by Google

35. If the terminal line of an angle lies between OX and OY, the angle is said to be in the *first quadrant;* if between OY and OX', in the *second quadrant;* between OX' and OY', in the *third quadrant;* between OY' and OX, in the *fourth quadrant.*

Thus, any positive angle between 0° and 90°, or between 360° and 450°, or any negative angle between $-270°$ and $-360°$, is in the first quadrant.

Any positive angle between 90° and 180°, or between 450° and 540°, or any negative angle between $-180°$ and $-270°$, is in the second quadrant.

36. We observe, by inspection of the results in Art. 33, the following points in regard to the algebraic signs of the trigonometric functions in the different quadrants:

For any angle in the first quadrant all the functions are positive.

In the second quadrant, the sine and cosecant are positive, and the cosine, tangent, cotangent, and secant are negative.

In the third quadrant, the tangent and cotangent are positive, and the sine, cosine, secant, and cosecant are negative.

In the fourth quadrant, the cosine and secant are positive, and the sine, tangent, cotangent, and cosecant are negative.

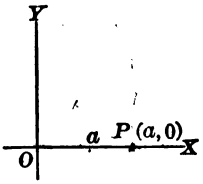
37. It is customary to express the foregoing principles in tabular form, as follows:

Functions.	*First Quad.*	*Second Quad.*	*Third Quad.*	*Fourth Quad.*
Sine and cosecant . . .	+	+	−	−
Cosine and secant . . .	+	−	−	+
Tangent and cotangent .	+	−	+	−

Digitized by Google

FUNCTIONS OF 0°, 90°, 180°, 270°, AND 360°.

38. *To find the functions of* 0° *and* 360°.



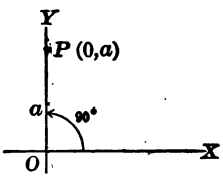
The terminal line of 0° coincides with the initial line OX.
Let P be a point on OX, such that $OP = a$.
Then by Art. 31, the co-ordinates of P are $(a, 0)$.
Whence by definition,

$$\sin 0° = \frac{0}{a} = 0. \qquad \tan 0° = \frac{0}{a} = 0. \qquad \sec 0° = \frac{a}{a} = 1.$$

$$\cos 0° = \frac{a}{a} = 1. \qquad \cot 0° = \frac{a}{0} = \infty. \qquad \csc 0° = \frac{a}{0} = \infty.$$

By Art. 34, the functions of 360° are the same as those of 0°.

39. *To find the functions of* 90°.



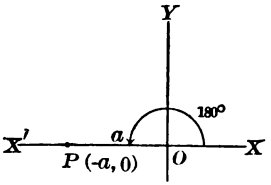
Let P be a point on OY, such that $OP = a$.
Then the co-ordinates of P are $(0, a)$.
Whence by definition,

$$\sin 90° = \frac{a}{a} = 1. \qquad \tan 90° = \frac{a}{0} = \infty. \qquad \sec 90° = \frac{a}{0} = \infty.$$

$$\cos 90° = \frac{0}{a} = 0. \qquad \cot 90° = \frac{0}{a} = 0. \qquad \csc 90° = \frac{a}{a} = 1.$$

Digitized by Google

40. *To find the functions of* 180°.



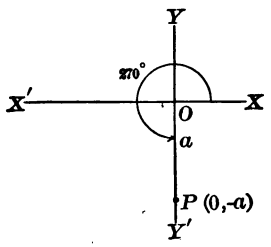
Let P be a point on OX', such that $OP = a$.
Then the co-ordinates of P are $(-a, 0)$.
Whence by definition,

$$\sin 180° = \frac{0}{-a} = 0. \qquad \cos 180° = \frac{-a}{a} = -1.$$

$$\tan 180° = \frac{0}{-a} = 0. \qquad \cot 180° = \frac{-a}{0} = \infty.$$

$$\sec 180° = \frac{a}{-a} = -1. \qquad \csc 180° = \frac{a}{0} = \infty.$$

41. *To find the functions of* 270°.



Let P be a point on OY', such that $OP = a$.
Then the co-ordinates of P are $(0, -a)$.
Whence by definition,

$$\sin 270° = \frac{-a}{a} = -1. \qquad \cos 270° = \frac{0}{a} = 0.$$

$$\tan 270° = \frac{-a}{0} = \infty. \qquad \cot 270° = \frac{0}{-a} = 0.$$

$$\sec 270° = \frac{a}{0} = \infty. \qquad \csc 270° = \frac{a}{-a} = -1.$$

Digitized by Google

Note. No absolute meaning can be attached to such results as $\cot 0^\circ = \infty$, $\tan 90^\circ = \infty$, etc. The equation $\cot 0^\circ = \infty$ merely signifies that as an angle approaches 0° as a limit, its cotangent increases without limit.

A similar interpretation must be given to the equations $\csc 0^\circ = \infty$, $\sec 90^\circ = \infty$, etc.

FUNCTIONS OF $(-A)$ IN TERMS OF THOSE OF A.

42. *To prove the formulæ,*

$$\left.\begin{array}{llll} \sin(-A) = -\sin A, & & \cos(-A) = & \cos A, \\ \tan(-A) = -\tan A, & & \cot(-A) = & -\cot A, \\ \sec(-A) = & \sec A, & \csc(-A) = & -\csc A, \end{array}\right\} \quad (9)$$

for any value of A.

There will be four cases, according as A is in the first, second, third, or fourth quadrant.

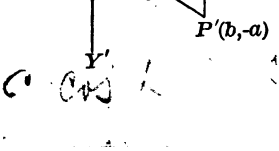
In each figure, let the positive angle XOP (indicated by the *full* arc) represent the angle A, and the negative angle XOP' (indicated by the *dotted* arc) the angle $(-A)$.

Draw PP' perpendicular to XX'; then the right triangles OPM and $OP'M$ have the side OM and the angle POM of one equal to the side OM and the angle $P'OM$ of the other, and are equal.

Whence, $PM = P'M$ and $OP = OP'$.

Let $PM = P'M = a$, $OM = b$, and $OP = OP' = c$.

CASE I. *A in the first quadrant, $-A$ in the fourth.*



Digitized by Google

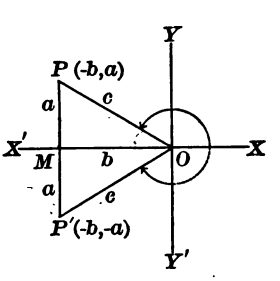
By definition, we have:

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$
$-A$	$-\frac{a}{c}$	$\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$\frac{c}{b}$	$-\frac{c}{a}$

It is evident from the above that the cosine and secant of $-A$ are equal respectively to the cosine and secant of A, while the sine, tangent, cotangent, and cosecant of $-A$ are equal respectively to the negatives of the sine, tangent, cotangent, and cosecant of A.

Whence we obtain the formulæ (9).

CASE II. *A in the second quadrant, $-A$ in the third.*

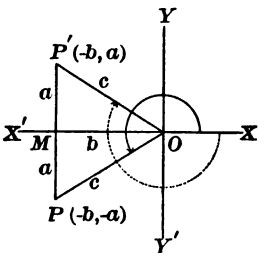


Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$-\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$-\frac{c}{b}$	$\frac{c}{a}$
$-A$	$-\frac{a}{c}$	$-\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$-\frac{c}{b}$	$-\frac{c}{a}$

Whence we obtain the formulæ (9) as before.

Digitized by Google

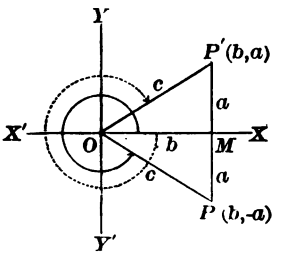
CASE III. *A in the third quadrant, $-A$ in the second.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$-\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$-\frac{c}{b}$	$-\frac{c}{a}$
$-A$	$\frac{a}{c}$	$-\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$-\frac{c}{b}$	$\frac{c}{a}$

Whence we obtain the formulæ (9) as before.

CASE IV. *A in the fourth quadrant, $-A$ in the first.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$\frac{c}{b}$	$-\frac{c}{a}$
$-A$	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$

Whence we obtain the formulæ (9) as before.

Digitized by Google

43. The results of Art. 42 may be stated as follows:

The sine, cosine, tangent, cotangent, secant, and cosecant of a negative angle are equal respectively to the negative sine, the cosine, the negative tangent, the negative cotangent, the secant, and the negative cosecant, of the absolute value of the angle.

FUNCTIONS OF $(90° + A)$ IN TERMS OF THOSE OF A.

44. *To prove the formulæ,*

$$\left.\begin{aligned} &\sin(90°+A) = \cos A, && \cos(90°+A) = -\sin A,\\ &\tan(90°+A) = -\cot A, && \cot(90°+A) = -\tan A,\\ &\sec(90°+A) = -\csc A, && \csc(90°+A) = \sec A, \end{aligned}\right\} \quad (10)$$

for any value of A.

There will be four cases, according as A is in the first, second, third, or fourth quadrant.

In each figure, let the positive angle XOP (indicated by the *full* arc) represent the angle A, and the positive angle XOP' (indicated by the *dotted* arc) the angle $(90° + A)$.

Lay off $OP = OP'$, and draw PM and $P'M'$ perpendicular to XX'.

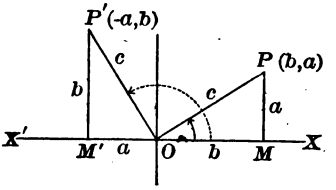
Since OP and OM are perpendicular to OP' and $P'M'$ respectively, the angles POM and $OP'M'$ are equal.

Then the right triangles OPM and $OP'M'$ have the hypotenuse OP and the angle POM of one equal to the hypotenuse OP' and the angle $OP'M'$ of the other, and are equal.

Whence, $PM = OM'$ and $OM = P'M'$.

Let $PM = OM' = a$, $OM = P'M' = b$, and $OP = OP' = c$.

CASE I. *A in the first quadrant, $90° + A$ in the second.*



Digitized by Google

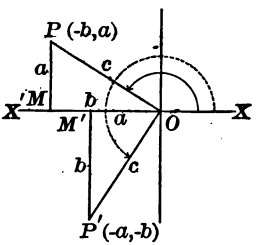
By definition, we have:

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$
$90°+A$	$\frac{b}{c}$	$-\frac{a}{c}$	$-\frac{b}{a}$	$-\frac{a}{b}$	$-\frac{c}{a}$	$\frac{c}{b}$

It is evident from the above that the sine and cosecant of $90° + A$ are equal respectively to the cosine and secant of A, while the cosine, tangent, cotangent, and secant of $90° + A$ are equal respectively to the negatives of the sine, cotangent, tangent, and cosecant of A.

Whence we obtain the formulæ (**10**).

Case II. *A in the second quadrant, $90° + A$ in the third.*

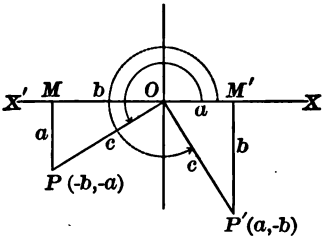


Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$-\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$-\frac{c}{b}$	$\frac{c}{a}$
$90°+A$	$-\frac{b}{c}$	$-\frac{a}{c}$	$\frac{b}{a}$	$\frac{a}{b}$	$-\frac{c}{a}$	$-\frac{c}{b}$

Whence we obtain the formulæ (**10**) as before.

Digitized by Google

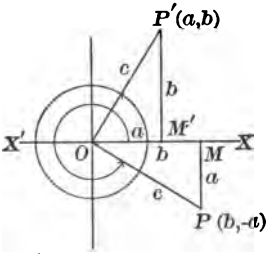
CASE III. *A in the third quadrant*, $90° + A$ *in the fourth.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$-\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$-\frac{c}{b}$	$-\frac{c}{a}$
$90°+A$	$-\frac{b}{c}$	$\frac{a}{c}$	$-\frac{b}{a}$	$-\frac{a}{b}$	$\frac{c}{a}$	$-\frac{c}{b}$

Whence we obtain the formulæ **(10)** as before.

CASE IV. *A in the fourth quadrant*, $90° + A$ *in the first.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$\frac{c}{b}$	$-\frac{c}{a}$
$90°+A$	$\frac{b}{c}$	$\frac{a}{c}$	$\frac{b}{a}$	$\frac{a}{b}$	$\frac{c}{a}$	$\frac{c}{b}$

Whence we obtain the formulæ **(10)** as before.

Digitized by Google

45. The results of Art. 44 may be stated in the following form:

The sine, cosine, tangent, cotangent, secant, and cosecant of any angle are equal respectively to the cosine, the negative sine, the negative cotangent, the negative tangent, the negative cosecant, and the secant, of an angle 90° less.

46. *To find the values of the functions of* $90° - A$ *in terms of those of* A.

By Arts. 45 and 42, we have:

$$\begin{aligned}
\sin(90°-A) &= \cos(-A) = \cos A.\\
\cos(90°-A) &= -\sin(-A) = \sin A.\\
\tan(90°-A) &= -\cot(-A) = \cot A.\\
\cot(90°-A) &= -\tan(-A) = \tan A.\\
\sec(90°-A) &= -\csc(-A) = \csc A.\\
\csc(90°-A) &= \sec(-A) = \sec A.
\end{aligned}$$

We have thus proved the formulæ of Art. 14 for any value of A.

47. *To find the values of the functions of* $180° - A$ *in terms of those of* A.

By Arts. 45 and 46 we have:

$$\begin{aligned}
\sin(180°-A) &= \cos(90°-A) = \sin A.\\
\cos(180°-A) &= -\sin(90°-A) = -\cos A.\\
\tan(180°-A) &= -\cot(90°-A) = -\tan A.\\
\cot(180°-A) &= -\tan(90°-A) = -\cot A.\\
\sec(180°-A) &= -\csc(90°-A) = -\sec A.\\
\csc(180°-A) &= \sec(90°-A) = \csc A.
\end{aligned}$$

Since the angles A and $180° - A$ are supplements of each other, these formulæ express the values of the functions of the supplement of an angle in terms of those of the angle itself.

Digitized by Google

48. *To find the values of the functions of* $180° + A$ *in terms of those of* A.

By Arts. 45 and 44, we have:

$$\sin(180°+A) = \cos(90°+A) = -\sin A.$$
$$\cos(180°+A) = -\sin(90°+A) = -\cos A.$$
$$\tan(180°+A) = -\cot(90°+A) = \tan A.$$
$$\cot(180°+A) = -\tan(90°+A) = \cot A.$$
$$\sec(180°+A) = -\csc(90°+A) = -\sec A.$$
$$\csc(180°+A) = \sec(90°+A) = -\csc A.$$

49. *To find the values of the functions of* $270° - A$ *in terms of those of* A.

By Arts. 45 and 47, we have:

$$\sin(270°-A) = \cos(180°-A) = -\cos A.$$
$$\cos(270°-A) = -\sin(180°-A) = -\sin A.$$
$$\tan(270°-A) = -\cot(180°-A) = \cot A.$$
$$\cot(270°-A) = -\tan(180°-A) = \tan A.$$
$$\sec(270°-A) = -\csc(180°-A) = -\csc A.$$
$$\csc(270°-A) = \sec(180°-A) = -\sec A.$$

50. *To find the values of the functions of* $270° + A$ *in terms of those of* A.

By Arts. 45 and 48, we have:

$$\sin(270°+A) = \cos(180°+A) = -\cos A.$$
$$\cos(270°+A) = -\sin(180°+A) = \sin A.$$
$$\tan(270°+A) = -\cot(180°+A) = -\cot A.$$
$$\cot(270°+A) = -\tan(180°+A) = -\tan A.$$
$$\sec(270°+A) = -\csc(180°+A) = \csc A.$$
$$\csc(270°+A) = \sec(180°+A) = -\sec A.$$

Digitized by Google

51. *To find the values of the functions of* $360°-A$ *and* $360°+A$ *in terms of those of* A.

By Art. 34, the functions of $360°-A$ are the same as those of $-A$, and the functions of $360°+A$ are the same as those of A.

Whence by Art. 42,

$\sin(360°-A) = -\sin A.$ $\quad$ $\cos(360°-A) = \cos A.$

$\tan(360°-A) = -\tan A.$ $\quad$ $\cot(360°-A) = -\cot A.$

$\sec(360°-A) = \sec A.$ $\quad$ $\csc(360°-A) = -\csc A.$

And,

$\sin(360°+A) = \sin A.$ $\quad$ $\cos(360°+A) = \cos A.$

$\tan(360°+A) = \tan A.$ $\quad$ $\cot(360°+A) = \cot A.$

$\sec(360°+A) = \sec A.$ $\quad$ $\csc(360°+A) = \csc A.$

52. The operation illustrated in Art. 51 may be extended indefinitely; we should find for the angles $450°-A$, $810°-A$, $-270°-A$, etc., the same results as for $90°-A$; and so on.

53. The results of Arts. 42 to 52 may be expressed in the following rule, which is derived by inspection from the formulæ of Arts. 42 to 51:

Any function of 0°, *or an* EVEN *multiple of* 90°, *plus or minus* A, *is the* SAME *function of* A; *and any function of an* ODD *multiple of* 90°, *plus or minus* A, *is the* COMPLEMENTARY *function of* A.

To obtain the algebraic sign, regard A *as an acute angle, and apply the table of* Art. 37.

1. Required the value of $\sec(990°+A)$.

Since 990° is an odd multiple of 90°, the absolute value of the result is $\csc A$.

And if A is an acute angle, $990°+A$ is in the fourth quadrant, in which the sign of the secant is positive.

Hence, $\sec(990°+A) = \csc A.$

Digitized by Google

2. Required the value of $\tan(-180°-A)$.

Since 180° is an even multiple of 90°, the absolute value of the result is $\tan A$.

And if A is an acute angle, $-180°-A$ is in the second quadrant, in which the sign of the tangent is negative.

Hence, $\tan(-180°-A) = -\tan A$.

EXAMPLES.

Find the values of the functions of the following angles in terms of the functions of A:

3. $450°-A$.	**7.** $630°-A$.	**11.** $-180°+A$.
4. $450°+A$.	**8.** $900°-A$.	**12.** $-180°-A$.
5. $540°-A$.	**9.** $-90°+A$.	**13.** $-270°+A$.
6. $540°+A$.	**10.** $-90°-A$.	**14.** $-720°+A$.

54. The functions of any positive or negative angle whatever may be expressed in terms of the functions of an angle between 0° and 90°.

1. Express $\sin 317°$ as a function of an angle between 0° and 90°.

We have by the rule of Art. 53,

$$\sin 317° = \sin(270°+47°) = -\cos 47°,$$

or $$\sin 317° = \sin(360°-43°) = -\sin 43°.$$

EXAMPLES.

Express the following in terms of the functions of angles between 0° and 90°:

2. $\cos 152°$.	**4.** $\sec(-77°)$.	**6.** $\cot(-129°)$.
3. $\tan 522°$.	**5.** $\csc 230°$.	**7.** $\sin 865°$.

Express the following in terms of the functions of angles between 0° and 45°:

8. $\cot 83°$.	**10.** $\sec 165°$.	**12.** $\tan 520°$.
9. $\sin(-50°)$.	**11.** $\cos(-303°)$.	**13.** $\csc 768°$.

Digitized by Google

55. The following table gives, for convenient reference, the functions of the angles 0°, 30°, 45°, 60°, 90°, 120°, etc.

The values of the functions of 0°, 30°, 45°, 60°, 90°, 180°, 270°, and 360° have already been proved in Arts. 16, 17, 38, 39, 40, and 41; the others may be derived by aid of Art. 53, and are left as exercises for the student.

As an illustration, we will give the proof for cot 150°.

By the rule of Art. 53:

$\cot 150° = \cot(180° - 30°) = -\cot 30° = -\sqrt{3}$ (Art. 17).

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
180°	0	-1	0	∞	-1	∞
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
270°	-1	0	∞	0	∞	-1
300°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
360°	0	1	0	∞	1	∞

Digitized by Google

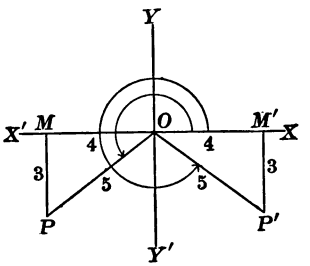
56. *Given the value of one of the functions of an angle, to find the values of the remaining functions.* (Compare Art. 15.)

1. Given $\sin A = -\frac{3}{5}$; required the values of the remaining functions of A.

The example may be solved by a method similar to that employed in Art. 15.

Since the sine is the ratio of the ordinate to the distance, we may regard the point of reference as having its ordinate equal to -3, and its distance equal to 5.

There are *two* such points, P and P', and consequently *two angles*, XOP and XOP', in the third and fourth quadrants respectively, either of which satisfies the given condition.



Then $OM = OM' = \sqrt{\overline{OP}^2 - \overline{PM}^2} = \sqrt{25-9} = 4$, and the co-ordinates of P are $(-4, -3)$, and of P', $(4, -3)$.

Whence, $\cos XOP = -\frac{4}{5}$, $\cos XOP' = \frac{4}{5}$,

$\tan XOP = \frac{3}{4}$, $\tan XOP' = -\frac{3}{4}$,

$\cot XOP = \frac{4}{3}$, $\cot XOP' = -\frac{4}{3}$,

$\sec XOP = -\frac{5}{4}$, $\sec XOP' = \frac{5}{4}$,

$\csc XOP = -\frac{5}{3}$, $\csc XOP' = -\frac{5}{3}$.

Digitized by Google

Thus the two solutions to the example are:

$$\cos A = \mp\frac{4}{5}, \quad \tan A = \pm\frac{3}{4}, \quad \cot A = \pm\frac{4}{3},$$

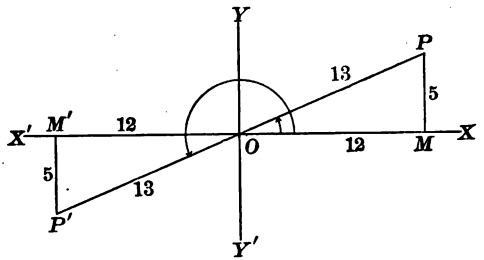
$$\sec A = \mp\frac{5}{4}, \quad \csc A = -\frac{5}{3},$$

where the upper signs refer to the angle XOP, and the lower signs to XOP'.

2. Given $\cot A = \frac{12}{5}$; required the values of the remaining functions of A.

The equation may be written $\cot A = \frac{-12}{-5}$.

We may then regard the point of reference as having its abscissa equal to 12 and its ordinate equal to 5, or as having its abscissa equal to -12 and its ordinate equal to -5; and there are consequently *two angles*, XOP and XOP', in the first and third quadrants respectively, either of which satisfies the given condition.



Then $OP = OP' = \sqrt{\overline{OM}^2 + \overline{PM}^2} = \sqrt{144 + 25} = 13.$

Whence, $\sin XOP = \frac{5}{13}$, $\sin XOP' = -\frac{5}{13}$,

$\cos XOP = \frac{12}{13}$, $\cos XOP' = -\frac{12}{13}$,

$\tan XOP = \frac{5}{12}$, $\tan XOP' = \frac{5}{12}$,

$\sec XOP = \frac{13}{12}$, $\sec XOP' = -\frac{13}{12}$,

$\csc XOP = \frac{13}{5}$, $\csc XOP' = -\frac{13}{5}$.

Digitized by Google

Thus the two solutions are,

$$\sin A = \pm\frac{5}{13}, \quad \cos A = \pm\frac{12}{13}, \quad \tan A = \frac{5}{12},$$

$$\sec A = \pm\frac{13}{12}, \quad \csc A = \pm\frac{13}{5}.$$

EXAMPLES.

In each case find the values of the remaining functions:

3. $\sin A = \frac{1}{4}$.

4. $\cot A = 2$.

5. $\csc A = -\frac{3}{2}$.

6. $\tan A = -\frac{8}{15}$.

7. $\sec A = \frac{4}{3}$.

8. $\cos A = -\frac{1}{2}$.

9. $\csc A = \sqrt{2}$.

10. $\tan A = 2\sqrt{2}$.

11. $\cos A = -\frac{a}{b}$.

12. $\sin A = x$.

13. $\cot A = \frac{1}{x}$.

14. $\sec A = \frac{\sqrt{a^2+b^2}}{a}$.

PROOFS OF THE FUNDAMENTAL FORMULÆ FOR ANY ANGLE.

57. We have from the general definitions of Art. 32,

$$\sin A \times \csc A = \frac{\text{ordinate}}{\text{distance}} \times \frac{\text{distance}}{\text{ordinate}} = 1.$$

$$\cos A \times \sec A = \frac{\text{abscissa}}{\text{distance}} \times \frac{\text{distance}}{\text{abscissa}} = 1.$$

$$\tan A \times \cot A = \frac{\text{ordinate}}{\text{abscissa}} \times \frac{\text{abscissa}}{\text{ordinate}} = 1.$$

Whence we obtain, as in Art. 18,

$$\sin A = \frac{1}{\csc A}, \quad \tan A = \frac{1}{\cot A}, \quad \sec A = \frac{1}{\cos A},$$

$$\cos A = \frac{1}{\sec A}, \quad \cot A = \frac{1}{\tan A}, \quad \csc A = \frac{1}{\sin A}.$$

Digitized by Google

58. *To prove the formulæ,*

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A, \quad \csc^2 A = 1 + \cot^2 A,$$

for any value of A.

Since the distance of the point of reference is the hypotenuse of a right triangle whose sides are equal to the absolute values of the abscissa and ordinate, we have by Geometry,

$$(\text{ordinate})^2 + (\text{abscissa})^2 = (\text{distance})^2.$$

This may be written in the forms,

$$\left(\frac{\text{ordinate}}{\text{distance}}\right)^2 + \left(\frac{\text{abscissa}}{\text{distance}}\right)^2 = 1,$$

$$\left(\frac{\text{distance}}{\text{abscissa}}\right)^2 = 1 + \left(\frac{\text{ordinate}}{\text{abscissa}}\right)^2,$$

and

$$\left(\frac{\text{distance}}{\text{ordinate}}\right)^2 = 1 + \left(\frac{\text{abscissa}}{\text{ordinate}}\right)^2.$$

Hence for any value of A, we have

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A, \quad \csc^2 A = 1 + \cot^2 A.$$

59. *To prove the formulæ*

$$\tan A = \frac{\sin A}{\cos A} \text{ and } \cot A = \frac{\cos A}{\sin A},$$

for any value of A.

By definition,

$$\tan A = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\dfrac{\text{ordinate}}{\text{distance}}}{\dfrac{\text{abscissa}}{\text{distance}}} = \frac{\sin A}{\cos A},$$

and

$$\cot A = \frac{\text{abscissa}}{\text{ordinate}} = \frac{\dfrac{\text{abscissa}}{\text{distance}}}{\dfrac{\text{ordinate}}{\text{distance}}} = \frac{\cos A}{\sin A},$$

Digitized by Google

IV. MISCELLANEOUS THEOREMS.

TO EXPRESS EACH OF THE SIX PRINCIPAL FUNCTIONS IN TERMS OF THE OTHER FIVE.

60. The following table expresses the value of each of the six principal functions in terms of the other five:

sin	$\cdots$	$\sqrt{1-\cos^2}$	$\frac{\tan}{\sqrt{1+\tan^2}}$	$\frac{1}{\sqrt{1+\cot^2}}$	$\frac{\sqrt{\sec^2-1}}{\sec}$	$\frac{1}{\csc}$
cos	$\sqrt{1-\sin^2}$	$\cdots$	$\frac{1}{\sqrt{1+\tan^2}}$	$\frac{\cot}{\sqrt{1+\cot^2}}$	$\frac{1}{\sec}$	$\frac{\sqrt{\csc^2-1}}{\csc}$
tan	$\frac{\sin}{\sqrt{1-\sin^2}}$	$\frac{\sqrt{1-\cos^2}}{\cos}$	$\cdots$	$\frac{1}{\cot}$	$\sqrt{\sec^2-1}$	$\frac{1}{\sqrt{\csc^2-1}}$
cot	$\frac{\sqrt{1-\sin^2}}{\sin}$	$\frac{\cos}{\sqrt{1-\cos^2}}$	$\frac{1}{\tan}$	$\cdots$	$\frac{1}{\sqrt{\sec^2-1}}$	$\sqrt{\csc^2-1}$
sec	$\frac{1}{\sqrt{1-\sin^2}}$	$\frac{1}{\cos}$	$\sqrt{1+\tan^2}$	$\frac{\sqrt{1+\cot^2}}{\cot}$	$\cdots$	$\frac{\csc}{\sqrt{\csc^2-1}}$
csc	$\frac{1}{\sin}$	$\frac{1}{\sqrt{1-\cos^2}}$	$\frac{\sqrt{1+\tan^2}}{\tan}$	$\sqrt{1+\cot^2}$	$\frac{\sec}{\sqrt{\sec^2-1}}$	$\cdots$

The reciprocal forms were proved in Art. 57; the others may be derived by aid of Arts. 57, 58, and 59, and are left as exercises for the student.

As an illustration, we will give a proof of the formula

$$\cos A = \frac{\sqrt{\csc^2 A - 1}}{\csc A}.$$

By Art. 58, $\cos A = \sqrt{1-\sin^2 A}$

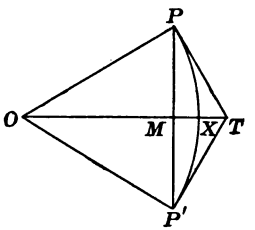
$$= \sqrt{1-\frac{1}{\csc^2 A}} = \frac{\sqrt{\csc^2 A - 1}}{\csc A}.$$

Digitized by Google

LIMITING VALUES OF $\frac{\sin x}{x}$ AND $\frac{\tan x}{x}$.

61. *To find the limiting values of the fractions* $\frac{\sin x}{x}$ *and* $\frac{\tan x}{x}$, *as* x *approaches the limit* 0.

Note. We suppose x to be expressed in *circular measure* (Art. 4).



Let $OPXP'$ be a circular sector; draw PT and $P'T$ tangent to the arc at P and P', and join OT and PP'.

By Geometry, $PT = P'T$.

Then OT is perpendicular to PP' at its middle point M, and bisects the arc PP' at X.

Let $\angle XOP = \angle XOP' = x$.

By Geometry, arc $PP' >$ chord PP', and $< PTP'$.

Whence, arc $PX > PM$, and $< PT$.

Therefore, $$\frac{\text{arc } PX}{OP} > \frac{PM}{OP}, \text{ and } < \frac{PT}{OP},$$

or, by Art. 5, circ. meas. $x > \sin x$, and $< \tan x$. (A)

Representing the circular measure of x by x simply, and dividing through by $\sin x$, we have

$$\frac{x}{\sin x} > 1, \text{ and } < \frac{\tan x}{\sin x} \text{ or } \frac{1}{\cos x}.$$

That is, $$\frac{\sin x}{x} < 1, \text{ and } > \cos x.$$

Digitized by Google

But as x approaches the limit 0, $\cos x$ approaches the limit 1 (Art. 38).

Hence, $\frac{\sin x}{x}$ approaches the limit 1 as x approaches 0.

Again, $$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \frac{\sin x}{x} \times \frac{1}{\cos x}.$$

But each factor approaches the limit 1 as x approaches 0.

Hence, $\frac{\tan x}{x}$ approaches the limit 1 as x approaches 0.

LINE VALUES OF THE TRIGONOMETRIC FUNCTIONS.

62. Let AOB be any angle.

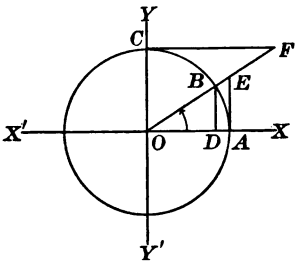


FIG. 1.

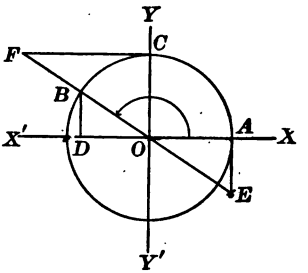


FIG. 2.

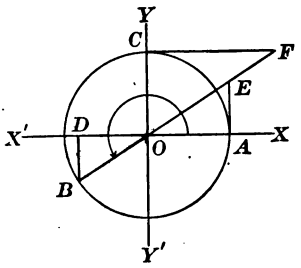


FIG. 3.

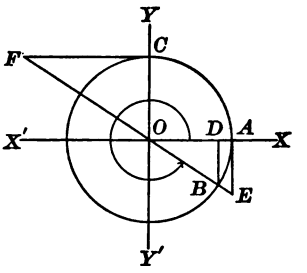


FIG. 4.

With O as a centre, and a radius equal to 1, describe the circle AB; draw BD and AE perpendicular to XX', and CF perpendicular to YY'.

Then by Art. 32, the functions of AOB are:

Digitized by Google

	Sin.	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
Fig. 1	$\frac{BD}{OB}$	$\frac{OD}{OB}$	$\frac{BD}{OD}$	$\frac{OD}{BD}$	$\frac{OB}{OD}$	$\frac{OB}{BD}$
Fig. 2	$\frac{BD}{OB}$	$-\frac{OD}{OB}$	$-\frac{BD}{OD}$	$-\frac{OD}{BD}$	$-\frac{OB}{OD}$	$\frac{OB}{BD}$
Fig. 3	$-\frac{BD}{OB}$	$-\frac{OD}{OB}$	$\frac{BD}{OD}$	$\frac{OD}{BD}$	$-\frac{OB}{OD}$	$-\frac{OB}{BD}$
Fig. 4	$-\frac{BD}{OB}$	$\frac{OD}{OB}$	$-\frac{BD}{OD}$	$-\frac{OD}{BD}$	$\frac{OB}{OD}$	$-\frac{OB}{BD}$

But since the right triangles OBD, OEA, and OCF are similar, and $OA = OC = 1$, we have,

$$\frac{BD}{OD} = \frac{AE}{OA} = AE, \qquad \frac{OB}{OD} = \frac{OE}{OA} = OE,$$

$$\frac{OD}{BD} = \frac{CF}{OC} = CF, \qquad \frac{OB}{BD} = \frac{OF}{OC} = OF.$$

Whence, since $OB = 1$, the functions of AOB are:

	Sin.	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
Fig. 1	BD	OD	AE	CF	OE	OF
Fig. 2	BD	$-OD$	$-AE$	$-CF$	$-OE$	OF
Fig. 3	$-BD$	$-OD$	AE	CF	$-OE$	$-OF$
Fig. 4	$-BD$	OD	$-AE$	$-CF$	OE	$-OF$

That is, *if the radius of the circle is unity*,

The *sine* is the perpendicular drawn to XX' from the intersection of the circle with the terminal line.

The *cosine* is the line drawn from the centre to the foot of the sine.

The *tangent* is that portion of the geometrical tangent to the circle at its intersection with OX included between OX and the terminal line, produced if necessary.

Digitized by Google

The *cotangent* is that portion of the geometrical tangent to the circle at its intersection with OY included between OY and the terminal line, produced if necessary.

The *secant* is that portion of the terminal line, or terminal line produced, included between the centre and the tangent.

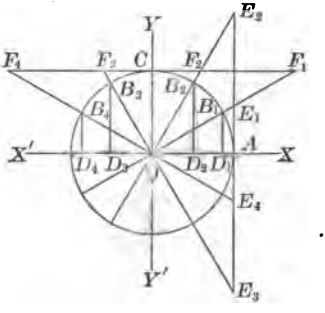
The *cosecant* is that portion of the terminal line, or terminal line produced, included between the centre and the cotangent.

And with regard to algebraic signs,

Sines and tangents measured *above* XX' are *positive*, and *below*, *negative;* cosines and cotangents measured to the *right* of YY' are *positive*, and to the *left*, *negative;* secants and cosecants measured on the terminal line itself are *positive*, and on the terminal line *produced*, *negative*.

63. The above are called the *line values* of the trigonometric functions. They simply *represent* the values of the functions when the radius is unity; that is, the *numerical value* of the sine of an angle is the same as the *number* which expresses the length of the perpendicular drawn to XX' from the intersection of the circle and terminal line.

64. *To trace the changes in the six principal trigonometric functions of an angle as the angle increases from* 0° *to* 360°.



Let the terminal line start from the position OA, and revolve about O as a pivot in the direction of OY, occupying successively the positions OB_1, OB_2, OC, OB_3, OB_4, etc.

Digitized by Google

Then since the sine of the angle commences with the value 0, and assumes in succession the values B_1D_1, B_2D_2, OC, B_3D_3, B_4D_4, etc. (Art. 60), it is evident that as the angle increases from 0° to 90°, the sine increases from 0 to 1; from 90° to 180°, it decreases from 1 to 0; from 180° to 270°, it decreases (algebraically) from 0 to -1; and from 270° to 360°, it increases from -1 to 0.

Since the cosine commences with the value OA, and assumes in succession the values OD_1, OD_2, 0, $-OD_3$, $-OD_4$, etc., from 0° to 90°, it decreases from 1 to 0; from 90° to 180°, it decreases from 0 to -1; from 180° to 270°, it increases from -1 to 0; and from 270° to 360°, it increases from 0 to 1.

Since the tangent commences with the value 0, and assumes in succession the values AE_1, AE_2, ∞, $-AE_3$, $-AE_4$, etc., from 0° to 90°, it increases from 0 to ∞; from 90° to 180°, it increases from $-\infty$ to 0; from 180° to 270°, it increases from 0 to ∞; and from 270° to 360°, it increases from $-\infty$ to 0.

Since the cotangent commences at ∞, and assumes in succession the values CF_1, CF_2, 0, $-CF_3$, $-CF_4$, etc., from 0° to 90°, it decreases from ∞ to 0; from 90° to 180°, it decreases from 0 to $-\infty$; from 180° to 270°, it decreases from ∞ to 0; and from 270° to 360°, it decreases from 0 to $-\infty$.

Since the secant commences at OA, and assumes in succession the values OE_1, OE_2, ∞, $-OE_3$, $-OE_4$, etc., from 0° to 90°, it increases from 1 to ∞; from 90° to 180°, it increases from $-\infty$ to -1; from 180° to 270°, it decreases from -1 to $-\infty$; and from 270° to 360°, it decreases from ∞ to 1.

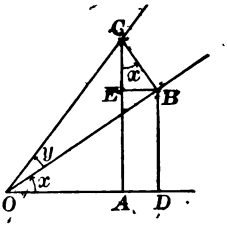
Since the cosecant commences at ∞, and assumes in succession the values OF_1, OF_2, OC, OF_3, OF_4, etc., from 0° to 90°, it decreases from ∞ to 1; from 90° to 180°, it increases from 1 to ∞; from 180° to 270°, it increases from $-\infty$ to -1; and from 270° to 360°, it decreases from -1 to $-\infty$.

Note. Wherever the symbol ∞ occurs in the foregoing discussion, it must be interpreted in accordance with the Note to Art. 41.

Digitized by Google

V. GENERAL FORMULÆ.

65. *To find the values of* $\sin(x+y)$ *and* $\cos(x+y)$ *in terms of the sines and cosines of* x *and* y.



Let AOB and BOC denote the angles x and y, respectively; then, $AOC = x + y$.

From any point C in OC draw CA and CB perpendicular to OA and OB; and draw BD and BE perpendicular to OA and AC.

Since EC and BC are perpendicular to OA and OB, the angles BCE and AOB are equal; that is, $BCE = x$.

We then have

$$\sin(x+y) = \frac{AC}{OC} = \frac{BD+CE}{OC} = \frac{BD}{OC} + \frac{CE}{OC}.$$

But
$$\frac{BD}{OC} = \frac{BD}{OB} \times \frac{OB}{OC} = \sin x \cos y,$$

and
$$\frac{CE}{OC} = \frac{CE}{BC} \times \frac{BC}{OC} = \cos x \sin y.$$

Whence, $\sin(x+y) = \sin x \cos y + \cos x \sin y.$ (11)

Again, $\cos(x+y) = \frac{OA}{OC} = \frac{OD-BE}{OC} = \frac{OD}{OC} - \frac{BE}{OC}.$

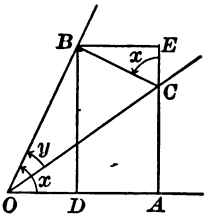
But
$$\frac{OD}{OC} = \frac{OD}{OB} \times \frac{OB}{OC} = \cos x \cos y,$$

and
$$\frac{BE}{OC} = \frac{BE}{BC} \times \frac{BC}{OC} = \sin x \sin y.$$

Whence, $\cos(x+y) = \cos x \cos y - \sin x \sin y.$ (12)

Digitized by Google

66. *To find the values of* $\sin(x-y)$ *and* $\cos(x-y)$ *in terms of the sines and cosines of* x *and* y.



Let AOB and BOC denote the angles x and y, respectively; then, $AOC = x - y$.

From any point C in OC draw CA and CB perpendicular to OA and OB; also, draw BD perpendicular to OA, and BE perpendicular to AC produced.

Since EC and BC are perpendicular to OA and OB, the angles BCE and AOB are equal; that is, $BCE = x$.

We then have

$$\sin(x-y) = \frac{AC}{OC} = \frac{BD-CE}{OC} = \frac{BD}{OC} - \frac{CE}{OC}.$$

But $$\frac{BD}{OC} = \frac{BD}{OB} \times \frac{OB}{OC} = \sin x \cos y,$$

and $$\frac{CE}{OC} = \frac{CE}{BC} \times \frac{BC}{OC} = \cos x \sin y.$$

Whence, $$\sin(x-y) = \sin x \cos y - \cos x \sin y. \qquad (13)$$

Again, $$\cos(x-y) = \frac{OA}{OC} = \frac{OD+BE}{OC} = \frac{OD}{OC} + \frac{BE}{OC}.$$

But $$\frac{OD}{OC} = \frac{OD}{OB} \times \frac{OB}{OC} = \cos x \cos y,$$

and $$\frac{BE}{OC} = \frac{BE}{BC} \times \frac{BC}{OC} = \sin x \sin y.$$

Whence, $$\cos(x-y) = \cos x \cos y + \sin x \sin y. \qquad (14)$$

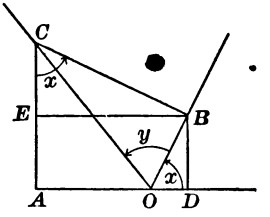
Digitized by Google

67. The fundamental formulæ of Arts. 65 and 66 are of great importance, and it is necessary to prove that they hold for all values of x and y.

It is obvious that the proof of Art. 65 is not general, for we have assumed in the construction of the figure that x and y are acute angles, and that $x+y$ is $< 90°$. Also, in Art. 66, we have taken x and y as acute angles, and $x > y$.

In order to prove the formulæ universally, we will first show that (**11**) and (**12**) hold for all values of x and y, and we can then give a general proof of (**13**) and (**14**).

68. We will first prove (**11**) and (**12**) when x and y are acute, and $x+y > 90°$.



Let DOB and BOC denote the angles x and y, respectively; then, $DOC = x+y$.

From any point C in OC draw CB perpendicular to OB, and CA perpendicular to OD produced; and draw BD and BE perpendicular to OD and AC.

Since EC and BC are perpendicular to OD and OB, the angles BCE and DOB are equal; that is, $BCE = x$.

We then have, by Art. 32,

$$\sin DOC = \frac{AC}{OC} = \frac{BD+CE}{OC} = \frac{BD}{OC} + \frac{CE}{OC}.$$

But
$$\frac{BD}{OC} = \frac{BD}{OB} \times \frac{OB}{OC} = \sin x \cos y,$$

and
$$\frac{CE}{OC} = \frac{CE}{BC} \times \frac{BC}{OC} = \cos x \sin y.$$

Whence, $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

Digitized by Google

Again, $$\cos DOC = \frac{-OA}{OC} = \frac{OD - BE}{OC} = \frac{OD}{OC} - \frac{BE}{OC}.$$

But $$\frac{OD}{OC} = \frac{OD}{OB} \times \frac{OB}{OC} = \cos x \cos y,$$

and $$\frac{BE}{OC} = \frac{BE}{BC} \times \frac{BC}{OC} = \sin x \sin y.$$

Whence, $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

69. We have thus proved (**11**) and (**12**) when x and y are *any two acute angles;* or, what is the same thing, when they are any two angles in the *first quadrant.*

Now let a and b be any assigned values of x and y for which (**11**) and (**12**) are true; then by Art. 44,

$$\sin[90° + (a+b)] = \cos(a+b)$$
$$= \cos a \cos b - \sin a \sin b, \quad \text{by (12)}; \quad (\mathbf{A})$$

and,

$$\cos[90° + (a+b)] = -\sin(a+b)$$
$$= -\sin a \cos b - \cos a \sin b, \text{ by (11)}. \quad (\mathbf{B})$$

But by Art. 44,

$$\cos a = \sin(90° + a), \qquad \cos b = \sin(90° + b),$$
$$-\sin a = \cos(90° + a), \qquad -\sin b = \cos(90° + b).$$

Hence (**A**) may be written in the forms,

$$\sin[(90° + a) + b] = \sin(90° + a)\cos b + \cos(90° + a)\sin b,$$
$$\sin[a + (90° + b)] = \sin a \cos(90° + b) + \cos a \sin(90° + b),$$

both of which are in accordance with (**11**).

And (**B**) may be written in the forms,

$$\cos[(90° + a) + b] = \cos(90° + a)\cos b - \sin(90° + a)\sin b,$$
$$\cos[a + (90° + b)] = \cos a \cos(90° + b) - \sin a \sin(90° + b),$$

both of which are in accordance with (**12**).

It follows from the above that if (**11**) and (**12**) hold for any assigned values of x and y, such as a and b, they also hold when either a or b is increased by 90°.

Digitized by Google

But they have been proved to hold when both x and y are in the first quadrant; hence they also hold when x is in the second quadrant and y in the first. And since they hold when x is in the second quadrant and y in the first, they also hold when x is in the third quadrant and y in the first; and so on.

Thus (**11**) and (**12**) are proved to hold for any values of x and y whatever, positive or negative.

70. We may now give a general proof of (**13**) and (**14**). By (**11**) and (**12**), we have

$$\begin{aligned}\sin(x-y) &= \sin[x+(-y)] \\ &= \sin x\cos(-y)+\cos x\sin(-y) \\ &= \sin x\cos y-\cos x\sin y \quad \text{(Art. 42)}.\end{aligned}$$

$$\begin{aligned}\text{And, } \cos(x-y) &= \cos[x+(-y)] \\ &= \cos x\cos(-y)-\sin x\sin(-y) \\ &= \cos x\cos y+\sin x\sin y \quad \text{(Art. 42)}.\end{aligned}$$

Hence (**13**) and (**14**) hold for all values of x and y, for the above proof depends on formulæ which have been shown to hold universally.

71. By Art. 20, we have

$$\begin{aligned}\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\ &= \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}, \text{ by (11) and (12)}.\end{aligned}$$

Dividing each term of the fraction by $\cos x\cos y$,

$$\begin{aligned}\tan(x+y) &= \frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}} \\ &= \frac{\tan x+\tan y}{1-\tan x\tan y}. \qquad (15)\end{aligned}$$

Digitized by Google

In like manner, we derive

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \qquad (16)$$

Again, by Art. 20, we have

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}$$

$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}, \text{ by (11) and (12).}$$

Dividing each term of the fraction by $\sin x \sin y$,

$$\cot(x+y) = \frac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\cos x \sin y}{\sin x \sin y}}$$

$$= \frac{\cot x \cot y - 1}{\cot y + \cot x}. \qquad (17)$$

In like manner, we derive

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}. \qquad (18)$$

72. From (11), (12), (13), and (14), we obtain

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b,$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b,$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b.$$

Whence, by addition and subtraction,

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b,$$
$$\sin(a+b) - \sin(a-b) = 2 \cos a \sin b,$$
$$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b,$$
$$\cos(a+b) - \cos(a-b) = -2 \sin a \sin b.$$

Digitized by Google

Let $a+b=x$, and $a-b=y$.

Then, $a=\frac{1}{2}(x+y)$, and $b=\frac{1}{2}(x-y)$.

Substituting these values, we have

$$\sin x+\sin y = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y), \quad (19)$$

$$\sin x-\sin y = 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y), \quad (20)$$

$$\cos x+\cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y), \quad (21)$$

$$\cos x-\cos y = -2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y). \quad (22)$$

73. Dividing (19) by (20), we obtain

$$\frac{\sin x+\sin y}{\sin x-\sin y}=\frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)}$$

$$=\tan\tfrac{1}{2}(x+y)\cot\tfrac{1}{2}(x-y)$$

$$=\frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)} \text{ (Art. 18).} \quad (23)$$

FUNCTIONS OF $2x$.

74. Putting $y=x$ in (11), we have

$$\sin 2x=\sin x\cos x+\cos x\sin x$$

$$=2\sin x\cos x. \quad (24)$$

Putting $y=x$ in (12),

$$\cos 2x=\cos^2 x-\sin^2 x. \quad (25)$$

Since $\cos^2 x=1-\sin^2 x$, and $\sin^2 x=1-\cos^2 x$ (Art. 19), we also have

$$\cos 2x=1-\sin^2 x-\sin^2 x \quad =1-2\sin^2 x, \quad (26)$$

and

$$\cos 2x=\cos^2 x-(1-\cos^2 x)=2\cos^2 x-1. \quad (27)$$

In like manner, from (15) and (17), we have

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}, \quad (28)$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x}. \quad (29)$$

Digitized by Google

FUNCTIONS OF $\frac{1}{2}x$.

75. From (**26**) and (**27**), we have

$$2\sin^2 x = 1 - \cos 2x, \text{ and } 2\cos^2 x = 1 + \cos 2x.$$

Writing x in place of $2x$, and therefore $\frac{1}{2}x$ in place of x,

$$2\sin^2 \tfrac{1}{2}x = 1 - \cos x, \text{ and } 2\cos^2 \tfrac{1}{2}x = 1 + \cos x. \quad (\mathbf{A})$$

Dividing by 2, and extracting the square root,

$$\sin \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}, \quad (\mathbf{30})$$

$$\cos \tfrac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}. \quad (\mathbf{31})$$

Dividing (**30**) by (**31**), we obtain

$$\tan \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{1+\cos x}}. \quad (\mathbf{32})$$

Multiplying the terms of the fraction under the radical sign first by $1 + \cos x$, and then by $1 - \cos x$, we have

$$\tan \tfrac{1}{2}x = \sqrt{\frac{1-\cos^2 x}{(1+\cos x)^2}}$$

$$= \sqrt{\frac{\sin^2 x}{(1+\cos x)^2}} = \frac{\sin x}{1+\cos x}, \quad (\mathbf{33})$$

and

$$\tan \tfrac{1}{2}x = \sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}}$$

$$= \sqrt{\frac{(1-\cos x)^2}{\sin^2 x}} = \frac{1-\cos x}{\sin x}. \quad (\mathbf{34})$$

And since the cotangent is the reciprocal of the tangent,

$$\cot \tfrac{1}{2}x = \frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}. \quad (\mathbf{35})$$

Note. The radical in each of the formulæ (**30**), (**31**), and (**32**) is to be taken as positive or negative according to the quadrant in which the angle $\frac{1}{2}x$ is situated (Art. 37).

Digitized by Google

INVERSE TRIGONOMETRIC FUNCTIONS.

76. The expression $\sin^{-1} y$, called the *inverse sine* of y, or the *anti-sine* of y, is used to denote the angle whose sine is equal to y.

Thus the fact that the sine of the angle x is equal to y may be expressed in either of the ways

$$\sin x = y, \text{ or } x = \sin^{-1} y.$$

In like manner, the expression $\cos^{-1} y$ signifies the angle whose cosine is equal to y; $\tan^{-1} y$, the angle whose tangent is equal to y, etc.

Note. The student must be careful not to confuse this notation with the *exponent* -1; the -1 power of $\sin x$ is expressed $(\sin x)^{-1}$, and not $\sin^{-1} x$.

77. By aid of the principles of Art. 76, any relation involving direct functions may be transformed into one involving inverse functions.

Take for example the formula

$$\sin (x + y) = \sin x \cos y + \cos x \sin y. \qquad (\text{A})$$

Let $\sin x = a$, and $\sin y = b$.

Then by Art. 76, $x = \sin^{-1} a$, and $y = \sin^{-1} b$.

Also by Art. 19, $\cos x = \sqrt{1 - \sin^2 x}$,

and $\cos y = \sqrt{1 - \sin^2 y}$.

Putting $\sin x = a$, and $\sin y = b$, we have

$$\cos x = \sqrt{1 - a^2}, \text{ and } \cos y = \sqrt{1 - b^2}.$$

Substituting these values in (A),

$$\sin (\sin^{-1} a + \sin^{-1} b) = a\sqrt{1 - b^2} + b\sqrt{1 - a^2}.$$

Whence by Art. 76,

$$\sin^{-1} a + \sin^{-1} b = \sin^{-1} (a\sqrt{1 - b^2} + b\sqrt{1 - a^2}).$$

Digitized by Google

EXERCISES.

78. 1. Prove the relation $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$.

By (25), $\cos 2x = \cos^2 x - \sin^2 x$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \text{ (Art. 19).}$$

Dividing each term of the fraction by $\cos^2 x$, we have

$$\cos 2x = \frac{1 - \dfrac{\sin^2 x}{\cos^2 x}}{1 + \dfrac{\sin^2 x}{\cos^2 x}} = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

2. Prove the relation $\dfrac{\sin 5x + \sin x}{\cos 5x + \cos x} = \tan 3x$.

By (19) and (21),

$$\frac{\sin 5x + \sin x}{\cos 5x + \cos x} = \frac{2\sin \frac{1}{2}(5x+x)\cos \frac{1}{2}(5x-x)}{2\cos \frac{1}{2}(5x+x)\cos \frac{1}{2}(5x-x)}$$

$$= \frac{\sin 3x}{\cos 3x} = \tan 3x.$$

Prove the following relations:

3. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

4. $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot x \cot y - 1}{\cot x \cot y + 1}$.

5. $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1 + \cot x \tan y}{\cot x + \tan y}$.

6. $\sin(45° + y) = \dfrac{\sin y + \cos y}{\sqrt{2}}$.

7. $\tan(60° - y) = \dfrac{\sqrt{3} - \tan y}{1 + \sqrt{3}\tan y}$.

8. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x+y)$.

Digitized by Google

9. $\dfrac{\sin x + \sin y}{\cos x - \cos y} = -\cot \tfrac{1}{2}(x-y)$.

10. $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$.

11. $\csc 2x = \tfrac{1}{2}\sec x \csc x$.

12. $\tan x + \cot x = \dfrac{2}{\sin 2x}$.

13. $\cot x - \tan x = 2\cot 2x$.

14. $\dfrac{(1+\tan x)^2 - (1-\tan x)^2}{(1+\tan x)^2 + (1-\tan x)^2} = \sin 2x$.

15. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.

16. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$.

17. $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$.

18. $\cos y + \cos(120° + y) + \cos(120° - y) = 0$.

19. $\sin A \sin(B-C) + \sin B \sin(C-A) + \sin C \sin(A-B) = 0$.

20. $\cos(A+B)\cos(A-B) + \cos(B+C)\cos(B-C) + \cos(C+A)\cos(C-A) = \cos 2A + \cos 2B + \cos 2C$.

21. $\dfrac{\cos x - \cos 3x}{\sin 3x - \sin x} = \tan 2x$.

22. $\dfrac{\cos 80° + \cos 20°}{\sin 80° - \sin 20°} = \sqrt{3}$.

By putting $x = x+y$ and $y = z$ in **(11)** and **(12)**, Art. 65, prove:

23. $\sin(x+y+z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$.

24. $\cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$.

Digitized by Google

By putting $x = 2x$ and $y = x$ in (11), (12), and (15), prove:

25. $\sin 3x = 3\sin x - 4\sin^3 x$.

26. $\cos 3x = 4\cos^3 x - 3\cos x$.

27. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

Prove the relations:

28. $\sin(2x + y) - 2\sin x \cos(x + y) = \sin y$.

29. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$.

30. $1 + \cos 2x \cos 2y = 2(\sin^2 x \sin^2 y + \cos^2 x \cos^2 y)$.

31. $1 + \tan x \tan 2x = \sec 2x$.

32. $\sin 4x = 4\sin x \cos x - 8\sin^3 x \cos x$.

33. $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$.

34. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$.

35. By putting $x = 45°$ and $y = 30°$ in (13) and (14), Art. 66, prove

$$\sin 15° = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}) = \cos 75°,$$
$$\cos 15° = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}) = \sin 75°.$$

36. By putting $x = 30°$ in (34) and (35), Art. 75, prove

$$\tan 15° = 2 - \sqrt{3} = \cot 75°,$$
$$\cot 15° = 2 + \sqrt{3} = \tan 75°.$$

37. By putting $x = 45°$ in (30) and (31), Art. 75, prove

$$\sin 22° 30' = \tfrac{1}{2}\sqrt{2 - \sqrt{2}}, \quad \cos 22° 30' = \tfrac{1}{2}\sqrt{2 + \sqrt{2}}.$$

38. By putting $x = 45°$ in (34) and (35), Art. 75, prove

$$\tan 22° 30' = \sqrt{2} - 1, \quad \cot 22° 30' = \sqrt{2} + 1.$$

Digitized by Google

VI. LOGARITHMS.

79. Every positive number may be expressed, exactly or approximately, as a power of 10; thus,

$$100 = 10^2;\ 13 = 10^{1.1139\ldots};\ \text{etc.}$$

When thus expressed, the corresponding exponent is called its *Logarithm to the base* 10; thus, 2 is the logarithm of 100 to the base 10, a relation which is written

$$\log_{10} 100 = 2, \text{ or simply } \log 100 = 2.$$

And in general, if $10^x = m$, then $x = \log m$.

80. Any positive number except unity may be taken as the base of a system of logarithms; thus, if $a^x = m$, then $x = \log_a m$.

Logarithms to the base 10 are called *Common Logarithms*, and are the only ones used in numerical computation.

If no base is expressed, the base 10 is understood.

81. We have by Algebra,

$$10^0 = 1, \qquad 10^{-1} = \tfrac{1}{10} = .1,$$
$$10^1 = 10, \qquad 10^{-2} = \tfrac{1}{100} = .01,$$
$$10^2 = 100, \qquad 10^{-3} = \tfrac{1}{1000} = .001, \text{ etc.}$$

Whence, by the definition of Art. 79,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$
$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$
$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

Note. The second form of the results for log .1, log .01, etc., is preferable in practice.

82. It is evident from Art. 81 that the logarithm of a number greater than 1 is positive, and that the logarithm of a number between 0 and 1 is negative.

Digitized by Google

83. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139$.

In this case the characteristic is 1, and the mantissa .1139.

84. It is evident from the first column of Art. 81 that the logarithm of any number between

1 and 10 is equal to 0 plus a decimal;
10 and 100 is equal to 1 plus a decimal;
100 and 1000 is equal to 2 plus a decimal; etc.

Hence, the characteristic of the logarithm of a number with *one* figure to the left of its decimal point, is 0; with *two* figures to the left of the decimal point, is 1; with *three* figures to the left of the decimal point, is 2; etc.

85. In like manner, from the second column of Art. 81, the logarithm of a decimal between

1 and .1 is equal to 9 plus a decimal -10;
.1 and .01 is equal to 8 plus a decimal -10;
.01 and .001 is equal to 7 plus a decimal -10; etc.

Hence, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure, is 9, with -10 after the mantissa; of a decimal with *one* cipher between its point and first figure, is 8, with -10 after the mantissa; of a decimal with *two* ciphers between its point and first figure, is 7, with -10 after the mantissa; etc.

86. For reasons which will be given hereafter, only the mantissa of the logarithm is given in tables of logarithms of numbers; the characteristic must be supplied by the reader.

The rules for characteristic are based on Arts. 84 and 85:

Digitized by Google

I. *If the number is greater than* 1, *the characteristic is* 1 *less than the number of places to the left of the decimal point.*

II. *If the number is between* 0 *and* 1, *subtract the number of ciphers between the decimal point and first significant figure from* 9, *writing* -10 *after the mantissa.*

Thus, characteristic of $\log 906328.5 = 5$;

characteristic of $\log .007023 = 7$, with -10 after the mantissa.

Note. Some writers, in dealing with the characteristics of negative logarithms, combine the two portions of the characteristic, writing the result as a *negative characteristic* before the mantissa.

Thus, instead of $7.6036 - 10$, the student will frequently find $\bar{3}.6036$, a minus sign being written over the characteristic to denote that it alone is negative, the mantissa being always positive.

PROPERTIES OF LOGARITHMS.

87. *In any system, the logarithm of unity is zero.*

For since $a^0 = 1$, we have $\log_a 1 = 0$ (Art. 79).

88. *In any system, the logarithm of the base itself is unity.*

For since $a^1 = a$, we have $\log_a a = 1$.

89. *In any system whose base is greater than unity, the logarithm of zero is minus infinity.*

For if a is > 1, we have $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$.

Whence by Art. 79, $\log_a 0 = -\infty$.

90. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{aligned} a^x &= m \\ a^y &= n \end{aligned}\right\}; \text{ whence by Art. 79, } \left\{\begin{aligned} x &= \log_a m, \\ y &= \log_a n. \end{aligned}\right.$$

Digitized by Google

Multiplying, we have

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $\log_a mn = x + y.$

Substituting the values of x and y, we have

$$\log_a mn = \log_a m + \log_a n.$$

In like manner, the theorem may be proved for the product of three or more factors.

91. By aid of the theorem of Art. 90, the logarithm of any composite number may be found when the logarithms of its factors are known.

1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned}\log 72 &= \log (2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \\ &= 3 \times \log 2 + 2 \times \log 3 \\ &= .9030 + .9542 = 1.8572.\end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$; find the values of the following:

2. $\log 6$.	**7.** $\log 21$.	**12.** $\log 98$.	**17.** $\log 135$.
3. $\log 14$.	**8.** $\log 63$.	**13.** $\log 105$.	**18.** $\log 168$.
4. $\log 8$.	**9.** $\log 56$.	**14.** $\log 112$.	**19.** $\log 147$.
5. $\log 12$.	**10.** $\log 84$.	**15.** $\log 144$.	**20.** $\log 375$.
6. $\log 15$.	**11.** $\log 45$.	**16.** $\log 216$.	**21.** $\log 343$.

92. *In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{aligned} a^x &= m \\ a^y &= n \end{aligned}\right\}; \text{ whence, } \left\{\begin{aligned} x &= \log_a m, \\ y &= \log_a n. \end{aligned}\right.$$

Digitized by Google

Dividing, we have $\frac{a^x}{a^y} = \frac{m}{n}$, or $a^{x-y} = \frac{m}{n}$.

Whence, $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$.

93. 1. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - .3010 = .6990.$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

2. $\log \frac{7}{3}$. **5.** $\log 35$. **8.** $\log \frac{42}{25}$. **11.** $\log 7\frac{1}{7}$.

3. $\log \frac{10}{7}$. **6.** $\log \frac{21}{16}$. **9.** $\log 175$. **12.** $\log \frac{35}{6}$.

4. $\log 3\frac{1}{3}$. **7.** $\log 125$. **10.** $\log 11\frac{1}{9}$. **13.** $\log 5\frac{4}{9}$.

94. *In any system, the logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation

$$a^x = m; \text{ whence, } x = \log_a m.$$

Raising both members to the pth power, we have

$$a^{px} = m^p; \text{ whence, } \log_a m^p = px = p \log_a m.$$

95. *In any system, the logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For, $\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$ (Art. 94).

96. 1. Given $\log 2 = .3010$; find the logarithm of $2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017.$$

Note. To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

Digitized by Google

2. Given $\log 3 = .4771$; find the logarithm of $\sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596.$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

3. $\log 3^{\frac{3}{5}}$. **7.** $\log 12^{\frac{2}{3}}$. **11.** $\log 15^{\frac{5}{6}}$. **15.** $\log \sqrt[6]{5}$.

4. $\log 2^{9}$. **8.** $\log 21^{\frac{1}{2}}$. **12.** $\log \sqrt{7}$. **16.** $\log \sqrt[4]{35}$.

5. $\log 7^{5}$. **9.** $\log 14^{4}$. **13.** $\log \sqrt[3]{3}$. **17.** $\log \sqrt[9]{98}$.

6. $\log 5^{\frac{1}{5}}$. **10.** $\log 25^{\frac{7}{3}}$. **14.** $\log \sqrt[7]{2}$. **18.** $\log \sqrt[12]{126}$.

19. Find the logarithm of $(2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By Art. 90, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$= \frac{1}{3} \log 2 + \frac{5}{4} \log 3$

$= .1003 + .5964 = .6967.$

Find the values of the following:

20. $\log \left(\frac{10}{3}\right)^{5}$. **22.** $\log (3^{\frac{1}{3}} \times 2^{\frac{3}{5}})$. **24.** $\log \sqrt{\frac{7}{3}}$. **26.** $\log \sqrt[3]{\frac{28}{5}}$.

21. $\log \frac{7^{\frac{3}{4}}}{5^{\frac{2}{3}}}$. **23.** $\log 3 \sqrt[4]{7}$. **25.** $\log \frac{\sqrt[3]{7}}{\sqrt[5]{2}}$. **27.** $\log \frac{\sqrt{42}}{10^{\frac{3}{4}}}$.

97. *In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.*

To illustrate, suppose that $\log 3.053 = .4847$.

Then,

$\log 30.53 = \log (10 \times 3.053) = \log 10 + \log 3.053$

$= 1 + .4847 = 1.4847;$

$\log 305.3 = \log (100 \times 3.053) = \log 100 + \log 3.053$

$= 2 + .4847 = 2.4847;$

$\log .03053 = \log (.01 \times 3.053) = \log .01 + \log 3.053$

$= 8 - 10 + .4847 = 8.4847 - 10;$ etc.

Digitized by Google

It is evident from the foregoing that if a number is multiplied or divided by any integral power of 10, thus producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

Thus, if $\log 3.053 = .4847$, then

$$\log 30.53 = 1.4847, \qquad \log .3053 = 9.4847 - 10,$$
$$\log 305.3 = 2.4847, \qquad \log .03053 = 8.4847 - 10,$$
$$\log 3053. = 3.4847, \qquad \log .003053 = 7.4847 - 10, \text{ etc.}$$

Note. The reason will now be seen for the statement made in Art. 86, that only the mantissæ are given in a table of logarithms of numbers. For, to find the logarithm of any number, we have only to take from the table the mantissa corresponding to its sequence of figures, and the characteristic may then be prefixed in accordance with the rules of Art. 86.

This property of logarithms is only enjoyed by the common system, and constitutes its superiority over others for the purposes of numerical computation.

98. 1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

$$\log 432 = \log (2^4 \times 3^3) = 4 \log 2 + 3 \log 3$$
$$= 1.2040 + 1.4313 = 2.6353.$$

Then by Art. 97, the *mantissa* of the result is .6353.

Whence by Art. 86, $\log .00432 = 7.6353 - 10$.

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

2. $\log 1.8$.
3. $\log 2.25$.
4. $\log .196$.
5. $\log .048$.
6. $\log 38.4$.
7. $\log .0054$.
8. $\log .000315$.
9. $\log 7350$.
10. $\log 4.05$.
11. $\log .448$.
12. $\log 302.4$.
13. $\log .06174$.
14. $\log (8.1)^7$.
15. $\log \sqrt[5]{9.6}$.
16. $\log (22.4)^{\frac{1}{4}}$.

Digitized by Google

99. *To prove the relation*

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

Assume the equations

$$\left.\begin{aligned} a^x &= m \\ b^y &= m \end{aligned}\right\}; \text{ whence, } \left\{\begin{aligned} x &= \log_a m, \\ y &= \log_b m. \end{aligned}\right.$$

From the assumed equations, we have

$$a^x = b^y, \text{ or } a^{\frac{x}{y}} = b.$$

Whence, $$\log_a b = \frac{x}{y}, \text{ or } y = \frac{x}{\log_a b}.$$

Substituting the values of x and y,

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

By aid of this relation, if the logarithm of a quantity m to a certain base a is known, its logarithm to any other base b may be found by dividing by the logarithm of b to the base a.

100. *To prove the relation*

$$\log_b a \times \log_a b = 1.$$

Putting $m = a$ in the result of Art. 99, we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \text{ (Art. 88).}$$

Whence, $$\log_b a \times \log_a b = 1.$$

APPLICATIONS.

101. The value of an arithmetical quantity, in which the operations indicated involve only multiplication, division, involution, or evolution, may be most conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

In operations with negative characteristics the rules of Algebra must be followed.

Digitized by Google

102. **1.** Find the value of $.0631 \times 7.208 \times .51272$.

By Art. 90, $\log (.0631 \times 7.208 \times .51272)$
$= \log .0631 + \log 7.208 + \log .51272.$

$\log .0631 = 8.8000 - 10$
$\log 7.208 = 0.8578$
$\log .51272 = 9.7099 - 10$

Adding, $\therefore$ log of result $= 19.3677 - 20$
$= 9.3677 - 10$ (see Note 1)

Number corresponding to $9.3677 - 10 = .23317$.

Note 1. If the sum is a negative logarithm, it should be reduced so that the negative portion of the characteristic may be -10.

Thus, $19.3677 - 20$ is reduced to $9.3677 - 10$.

2. Find the value of $\frac{336.8}{7984}$.

By Art. 92, $\log \frac{336.8}{7984} = \log 336.8 - \log 7984.$

$\log 336.8 = 12.5273 - 10$ (see Note 2)
$\log 7984 = 3.9022$

Subtracting, $\therefore$ log of result $= 8.6251 - 10$
Number corresponding $= .04218.$

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By Art. 94, $\log (.07396)^5 = 5 \times \log .07396.$

$\log .07396 = 8.8690 - 10$
5
$44.3450 - 50$
$= 4.3450 - 10$ (see Note 1)
$= \log .000002213.$

Digitized by Google

4. Find the value of $\sqrt[3]{.035063}$.

By Art. 95, $\log\sqrt[3]{.035063} = \frac{1}{3}\log .035063.$

$$\log .035063 = 8.5449 - 10$$

$$20. \qquad -20 \text{ (see Note 3)}$$

$$3\overline{)28.5449 - 30}$$

$$9.5150 - 10 = \log .3274.$$

Note 3. To divide a negative logarithm, add to both parts such a multiple of 10 as will make the negative portion of the characteristic exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, add 20 to both parts of the logarithm, giving the result $28.5449 - 30$. Dividing this by 3, the quotient is $9.5150 - 10$.

ARITHMETICAL COMPLEMENT.

103. The *Arithmetical Complement* of the logarithm of a number, or briefly the *Cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus, $$\operatorname{colog} 409 = \log\frac{1}{409} = \log 1 - \log 409.$$

$$\log 1 = 10. \qquad -10 \text{ (Note 2, Art. 102)}$$

$$\log 409 = \underline{2.6117}$$

$$\therefore \operatorname{colog} 409 = 7.3883 - 10.$$

Again, $$\operatorname{colog} .067 = \log\frac{1}{.067} = \log 1 - \log .067.$$

$$\log 1 = 10. \qquad -10$$

$$\log .067 = \underline{8.8261 - 10}$$

$$\therefore \operatorname{colog} .067 = 1.1739.$$

The following rule is evident from the above:

To find the cologarithm of a number, subtract its logarithm from $10 - 10$.

Note. The cologarithm may be obtained from the logarithm by subtracting the last *significant* figure from 10 and each of the others from 9, -10 being written after the result in the case of a positive logarithm.

Digitized by Google

104. *Example.* Find the value of $\frac{.51384}{8.709 \times .0946}$.

$$\log \frac{.51384}{8.709 \times .0946} = \log \left(.51384 \times \frac{1}{8.709} \times \frac{1}{.0946} \right)$$

$$= \log .51384 + \log \frac{1}{8.709} + \log \frac{1}{.0946}$$

$$= \log .51384 + \operatorname{colog} 8.709 + \operatorname{colog} .0946.$$

$$\log .51384 = 9.7109 - 10$$
$$\operatorname{colog} 8.709 = 9.0601 - 10$$
$$\operatorname{colog} .0946 = \underline{1.0241}$$
$$9.7951 - 10 = \log .6239.$$

It is evident from the above that the logarithm of a fraction is equal to the logarithm of the numerator *plus* the cologarithm of the denominator.

Or in general, to find the logarithm of a fraction whose terms are composed of factors,

Add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.

Note. The value of the above fraction may be found without using cologarithms, by the following formula:

$$\log \frac{.51384}{8.709 \times .0946} = \log .51384 - \log (8.709 \times .0946)$$

$$= \log .51384 - (\log 8.709 + \log .0946).$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

EXAMPLES.

105. Note. A *negative* quantity can have no common logarithm, as is evident from the definition of Art. 79. If negative quantities occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3, p. 65, the value of $721.3 \times (-3.0528)$ may be obtained by finding the value of 721.3×3.0528, and putting a negative sign before the result. See also Ex. 34, p. 66.

Digitized by Google

Find by logarithms the values of the following:

1. $9.238 \times .9152$.
2. $130.36 \times .08237$.
3. $721.3 \times (-3.0528)$.
4. $(-4.3264) \times (-.050377)$
5. $.27031 \times .042809$.
6. $(-.063165) \times 11.134$.
7. $\frac{401.8}{52.37}$.
8. $\frac{7.2321}{10.813}$.
9. $\frac{-.3384}{.08659}$.
10. $\frac{9.163}{.0051422}$.
11. $\frac{22518}{64327}$.
12. $\frac{.007514}{-.015822}$.
13. $\frac{3.3681}{12.853 \times .6349}$.
14. $\frac{15.008 \times (-.0843)}{.06376 \times 4.248}$.
15. $\frac{(-2563) \times .03442}{714.8 \times (-.511)}$.
16. $\frac{121.6 \times (-9.025)}{(-48.3) \times 3662 \times (-.0856)}$.
17. $(23.86)^3$.
18. $(.532)^8$.
19. $(-1.0246)^7$.
20. $(.09323)^5$.
21. $5^{\frac{2}{3}}$.
22. $(.8)^{\frac{2}{3}}$.
23. $(-3.16)^{\frac{4}{5}}$.
24. $(.021)^{\frac{5}{7}}$.
25. $\sqrt{2}$.
26. $\sqrt[4]{5}$.
27. $\sqrt[5]{-3}$.
28. $\sqrt{.4294}$.
29. $\sqrt[3]{.02305}$.
30. $\sqrt[8]{1000}$.
31. $\sqrt[7]{-.00951}$.
32. $\sqrt[5]{.0001011}$.

33. Find the value of $\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log \sqrt[3]{5} + \operatorname{colog} 3^{\frac{5}{6}}$$
$$= \log 2 + \tfrac{1}{3} \log 5 + \tfrac{5}{6} \operatorname{colog} 3.$$

$\log 2 = .3010$

$\log 5 = .6990$; divide by $3 = .2330$

$\operatorname{colog} 3 = 9.5229 - 10$; multiply by $\frac{5}{6} = 9.6024 - 10$

$.1364 = \log 1.3691.$

Digitized by Google

34. Find the value of $\sqrt[3]{\frac{-.03296}{7.962}}$.

$$\log \sqrt[3]{\frac{.03296}{7.962}} = \tfrac{1}{3}\log\frac{.03296}{7.962} = \tfrac{1}{3}(\log .03296 - \log 7.962).$$

$$\log .03296 = 8.5180 - 10$$
$$\log 7.962 = 0.9010$$
$$3\overline{)27.6170 - 30}$$
$$9.2057 - 10 = \log .16059.$$

Result, $-.16059$.

Find the values of the following:

35. $2^{\frac{1}{2}} \times 3^{\frac{2}{3}}$.

36. $\dfrac{3^{\frac{5}{6}}}{4^{\frac{2}{3}}}$.

37. $\dfrac{5^{\frac{3}{7}}}{(-10)^{\frac{2}{9}}}$.

38. $\left(\frac{6}{7}\right)^{\frac{5}{2}}$.

39. $\left(\frac{35}{113}\right)^{\frac{3}{5}}$.

40. $\left(\frac{.08726}{.1321}\right)^{\frac{5}{3}}$.

41. $\sqrt[3]{\frac{21}{13}}$.

42. $\sqrt[9]{-\frac{3}{7}}$.

43. $\sqrt[5]{\frac{2}{3}} \div \sqrt[3]{\frac{3}{5}}$.

44. $\sqrt[3]{2} \times \sqrt[5]{3} \times \sqrt[7]{.01}$.

45. $\sqrt[5]{\frac{3258}{49309}}$.

46. $\left(\frac{-31.63}{429}\right)^{\frac{3}{17}}$.

47. $\dfrac{100^{\frac{2}{3}}}{(.7325)^{\frac{3}{7}}}$.

48. $\dfrac{\sqrt[3]{.0001289}}{\sqrt[4]{.0008276}}$.

49. $\dfrac{(-.7469)^{\frac{5}{3}}}{-(.2345)^{\frac{7}{2}}}$.

50. $\dfrac{\sqrt[11]{.0073}}{(.68291)^{\frac{5}{2}}}$.

51. $\dfrac{\sqrt{5.955} \times \sqrt[3]{61.2}}{\sqrt[5]{298.54}}$.

52. $(538.2 \times .0005969)^{\frac{1}{3}}$.

53. $(18.9503)^{11} \times (-.1)^{14}$.

54. $\sqrt[6]{3734.9 \times .00001108}$.

55. $(2.6317)^{\frac{3}{4}} \times (.71272)^{\frac{2}{5}}$.

56. $\dfrac{\sqrt[3]{-.008193} \times (.06285)^{\frac{3}{2}}}{-.98342}$.

57. $\sqrt{.035} \times \sqrt[6]{.62667} \times \sqrt[3]{.0072103}$.

Digitized by Google

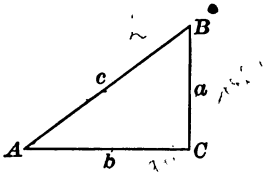
VII. SOLUTION OF RIGHT TRIANGLES.

106. The six *elements* of a triangle are its three sides and its three angles.

We know by Geometry that, in general, a triangle is completely determined when three of its elements are known, provided one of them is a side. The *solution* of a triangle is the process of computing the unknown from the given elements.

107. To solve a *right triangle, two* elements must be given in addition to the right angle, one of which must be a side.

The various cases which can occur may all be solved by aid of the following formulæ:



$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b},$$

$$\sin B = \frac{b}{c}, \qquad \cos B = \frac{a}{c}, \qquad \tan B = \frac{b}{a}.$$

CASE I. *When the given elements are a side and an angle.*

The proper formula for computing either of the remaining sides may be found by the following rule:

Take that function of the angle which involves the given side and the required side.

CASE II. *When both the given elements are sides.*

First calculate one of the angles by aid of either of the formulæ involving the given elements, and then compute the remaining side as in Case I.

Digitized by Google

EXAMPLES.

108. **1.** Given $c = 203.76$, $B = 21° 43'$. Find a and b.

In this case the formulæ to be used are

$$\cos B = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.$$

Whence, $a = c \cos B$, and $b = c \sin B$.

By logarithms, $\log a = \log c + \log \cos B$,

and $\log b = \log c + \log \sin B$.

$\log c = 2.3091$	$\log c = 2.3091$
$\log \cos B = 9.9681 - 10$	$\log \sin B = 9.5682 - 10$
$\log a = 2.2772$	$\log b = 1.8773$
$\therefore a = 189.3.$	$\therefore b = 75.38.$

2. Given $a = 13$, $A = 67° 7'$. Find b and c.

In this case, $\tan A = \frac{a}{b}$, and $\sin A = \frac{a}{c}$.

Whence, $b = \frac{a}{\tan A}$, and $c = \frac{a}{\sin A}$.

By logarithms, $\log b = \log a - \log \tan A$,

and $\log c = \log a - \log \sin A$.

$\log a = 1.1139$	$\log a = 1.1139$
$\log \tan A = 0.3746$	$\log \sin A = 9.9644 - 10$
$\log b = 0.7393$	$\log c = 1.1495$
$\therefore b = 5.486.$	$\therefore c = 14.11.$

3. Given $b = .1512$, $c = .3081$. Find A and a.

In this case, $\cos A = \frac{b}{c}$, and $\tan A = \frac{a}{b}$, or $a = b \tan A$.

Whence, $\log \cos A = \log b - \log c$,

and $\log a = \log b + \log \tan A$.

Digitized by Google

$$\log b = 9.1796 \qquad \log b = 9.1796$$
$$\log c = \underline{9.4887} \qquad \log \tan A = \underline{0.2493}$$
$$\log \cos A = 9.6909 \qquad \log a = 9.4289$$
$$\therefore A = 60° 36.4'. \qquad \therefore a = .2685.$$

Note. It is customary in practice to omit writing the -10 after the mantissa of a negative logarithm, as illustrated in Ex. 3.

109. In the Trigonometrical solution of a triangle by the method of Case II., it is necessary to first find one of the angles, and the remaining side may then be calculated.

It is possible however to obtain the third side directly, without first finding the angle, by Geometrical methods.

Thus in Ex. 3, Art. 108, we have by Geometry,

$$a^2 + b^2 = c^2,$$

Whence, $$a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)}.$$

By logarithms, $$\log a = \tfrac{1}{2}[\log (c+b) + \log (c-b)].$$

$$c + b = .4593; \ \log = 9.6621 - 10$$
$$c - b = .1569; \ \log = 9.1956 - 10$$
$$2)\overline{18.8577 - 20}$$
$$\log a = 9.4289 - 10$$
$$\therefore a = .2685, \text{ as before.}$$

If the given sides are a and b, the formula for c is $\sqrt{a^2 + b^2}$, which is not adapted to logarithmic computation.

In such a case it is usually shorter to proceed according to the rule of Art. 107.

EXAMPLES.

110. Solve the following right triangles:

1. Given $A = 43° 30'$, $c = 11.2$.
2. Given $B = 68° 50'$, $a = 729.3$.
3. Given $B = 62° 56'$, $b = 47.7$.
4. Given $a = .624$, $c = .91$.

Digitized by Google

5. Given $a = 5$, $b = 2$.
6. Given $A = 72° 7'$, $a = 83.4$.
7. Given $B = 32° 10'$, $c = .02728$.
8. Given $b = 2.887$, $c = 5.11$.
9. Given $A = 52° 41'$, $b = 4247$.
10. Given $a = 101$, $b = 116$.
11. Given $A = 43° 22'$, $a = 158.3$.
12. Given $A = 58° 39'$, $c = 35.73$.
13. Given $a = 204.2$, $c = 275.3$.
14. Given $B = 30°$, $b = 1.6438$.
15. Given $A = 22° 14'$, $b = 13.242$.
16. Given $B = 10° 51'$, $c = .7264$.
17. Given $a = 638.5$, $b = 501.2$.
18. Given $A = 78° 17'$, $a = 203.8$.
19. Given $b = .02497$, $c = .04792$.
20. Given $B = 2° 19' 30''$, $a = 1875.3$.
21. Given $a = 24.67$, $b = 33.02$.
22. Given $b = 1.4367$, $c = 3.4653$.
23. Given $B = 6° 12.3'$, $c = 37206$.
24. Given $A = 64° 1.3'$, $b = 200.05$.
25. Given $a = 340.06$, $b = 231.69$.
26. Given $a = 1.7087$, $c = 2.0008$.
27. Given $B = 21° 33' 51''$, $a = .82109$.
28. Given $A = 74° 0' 18''$, $c = 275.62$.
29. Given $B = 34° 14' 37''$, $b = 120.22$.
30. Given $a = 10.107$, $b = 17.303$.

Digitized by Google

Solve the following isosceles triangles, in which A and B are the equal angles, and a, b, and c denote the sides opposite the angles A, B, and C, respectively:

31. Given $A = 68° 57'$, $b = 35.091$.

32. Given $B = 27° 8'$, $c = 3.088$.

33. Given $C = 80° 47'$, $b = 2103.2$.

34. Given $a = 79.24$, $c = 106.62$.

35. Given $A = 70° 19'$, $c = .5623$.

36. Given $C = 151° 28'$, $c = 95.47$.

37. A regular pentagon is inscribed in a circle whose diameter is 24 inches. Find the length of its side.

38. At a distance of 100 feet from the base of a tower, the angle of elevation of its top is observed to be 38°. Find its height.

39. What is the angle of elevation of the sun when a tower whose height is 103.7 feet casts a shadow 167.3 feet in length?

40. If the diameter of a circle is 3268, find the angle at the centre subtended by an arc whose chord is 1027.

41. If the diameter of the earth is 7912 miles, what is the distance of the remotest point of the surface visible from the summit of a mountain $1\frac{1}{4}$ miles in height?

42. Find the length of the diagonal of a regular pentagon whose side is 7.028 inches.

43. What is the angle of elevation of a mountain-slope which rises 238 feet in a horizontal distance of one-eighth of a mile?

44. From the top of a lighthouse, 133 feet above the sea, the angle of depression of a buoy is observed to be 18° 25′. Required the horizontal distance of the buoy.

Digitized by Google

45. A ship is sailing due east at the rate of 7.8 miles an hour. A headland is observed to bear due north at 10.37 A.M., and 33° west of north at 12.43 P.M. Find the distance of the headland from each point of observation.

46. If a chord of 41.36 feet subtends an arc of 145° 37′, what is the radius of the circle?

47. The length of the side of a regular octagon is 12 inches. Find the radii of the inscribed and circumscribed circles.

48. How far from the foot of a pole 80 feet high must an observer stand, so that the angle of elevation of the top of the pole may be 10°?

49. If the diagonal of a regular pentagon is 32.83 inches, what is the radius of the circumscribed circle?

50. From the top of a tower, the angle of depression of the extremity of a horizontal base line, 1000 feet in length measured from the foot of the tower, is observed to be 21° 16′ 37″. Find the height of the tower.

51. If the radius of a circle is 723.29, what is the length of the chord which subtends an arc of 35° 13′?

52. A regular hexagon is circumscribed about a circle whose diameter is 10 inches. Find the length of its side.

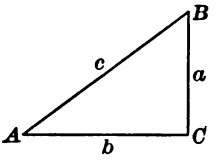
53. From the top of a lighthouse, 200 feet above the sea, the angles of depression of two boats in line with the lighthouse are observed to be 14° and 32° respectively. What is the distance between the boats?

54. A ship is sailing due east at a uniform rate of speed. At 7 A.M., a lighthouse is observed bearing due north, 10.32 miles distant, and at 7.30 A.M. it bears 18° 13′ west of north. Find the rate of sailing of the ship and the bearing of the lighthouse at 10 A.M.

Digitized by Google

FORMULÆ FOR THE AREA OF A RIGHT TRIANGLE.

111. CASE I. *Given the hypotenuse and an acute angle.*



Denoting the area by K, we have by Geometry,

$$2K = ab.$$

But by Art. 13, $a = c \sin A$,

and $b = c \cos A$.

Whence, $2K = c^2 \sin A \cos A = \frac{1}{2} c^2 \sin 2A$ (Art. 74).

That is, $4K = c^2 \sin 2A.$ **(36)**

In like manner,

$$4K = c^2 \sin 2B. \quad \textbf{(37)}$$

CASE II. *Given an angle and its opposite side.*

We have, $\cot A = \frac{b}{a}$, or $b = a \cot A$.

Whence, $2K = a \cdot a \cot A = a^2 \cot A.$ **(38)**

In like manner,

$$2K = b^2 \cot B. \quad \textbf{(39)}$$

CASE III. *Given an angle and its adjacent side.*

We have, $\cot A = \tan B$ (Art. 14).

Whence by **(38)**,

$$2K = a^2 \tan B. \quad \textbf{(40)}$$

In like manner,

$$2K = b^2 \tan A. \quad \textbf{(41)}$$

Digitized by Google

CASE IV. *Given the hypotenuse and another side.*

Since $a^2 + b^2 = c^2$, we have

$$2K = ab = a\sqrt{c^2 - a^2}$$

$$= a\sqrt{(c+a)(c-a)}. \qquad (42)$$

In like manner,

$$2K = b\sqrt{(c+b)(c-b)}. \qquad (43)$$

CASE V. *Given the two sides about the right angle.*

In this case, $2K = ab.$ (44)

EXAMPLES.

112. **1.** Given $c = 10.36$, $B = 75°$; find the area.

By (37), $4K = c^2 \sin 2B.$

Whence, $\log(4K) = 2\log c + \log \sin 2B.$

$\log c = 1.0153$; multiply by $2 = 2.0306$

$2B = 150°$; $\log \sin = \underline{9.6990 - 10}$

$\log(4K) = 1.7296$

$\therefore 4K = 53.65$, and $K = 13.41.$

Note. To find log sin 150°, take either log cos 60° or log sin 30°. (See page 10 of the explanation of the tables.)

Find the areas of the following triangles:

2. Given $A = 19° 36'$, $a = 22.17.$

3. Given $B = 24° 7'$, $a = .8213.$

4. Given $a = 149.31$, $b = 76.29.$

5. Given $b = .3056$, $c = .6601.$

6. Given $A = 30° 56' 20''$, $c = 192.9.$

7. Given $A = 58° 52'$, $b = .05207.$

8. Given $a = .932$, $c = 2.786.$

9. Given $B = 72° 25'$, $c = 27.283.$

10. Given $B = 29° 18' 15''$, $b = .33784.$

Digitized by Google

VIII. GENERAL PROPERTIES OF TRIANGLES.

113. *In any triangle, the sides are proportional to the sines of their opposite angles.*

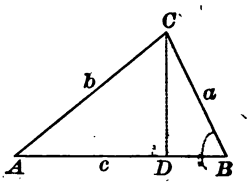


FIG. 1.

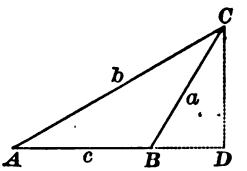


FIG. 2.

There will be two cases, according as the angles are all acute (Fig. 1), or one of them obtuse (Fig. 2).

In each case let CD be drawn perpendicular to AB.

Then in either figure, we have

$$\sin A = \frac{CD}{b}.$$

In Fig. 1, $$\sin B = \frac{CD}{a},$$

and in Fig. 2, $$\sin B = \sin (180° - CBD)$$

$$= \sin CBD \text{ (Art. 47)}$$

$$= \frac{CD}{a}.$$

Dividing these equations, we have in either case

$$\frac{\sin A}{\sin B} = \frac{\frac{CD}{b}}{\frac{CD}{a}} = \frac{a}{b}. \qquad (45)$$

Digitized by Google

In like manner, we may prove

$$\frac{\sin B}{\sin C} = \frac{b}{c}, \tag{46}$$

and

$$\frac{\sin C}{\sin A} = \frac{c}{a}. \tag{47}$$

The above results may be expressed more compactly as follows:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

114. *In any triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Formula (45) of Art. 113 may be put in the form

$$a : b = \sin A : \sin B.$$

Whence, by composition and division,

$$a + b : a - b = \sin A + \sin B : \sin A - \sin B,$$

or,

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

But by Art. 73,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

Whence,

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \tag{48}$$

In like manner, we may prove

$$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}, \tag{49}$$

and

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)}. \tag{50}$$

Digitized by Google

115. Since $A + B = 180° - C$, we have

$$\tan \tfrac{1}{2}(A + B) = \tan (90° - \tfrac{1}{2} C) = \cot \tfrac{1}{2} C \text{ (Art. 14).}$$

Thus formula (**48**) may be put in the form

$$\frac{a + b}{a - b} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A - B)}. \qquad (51)$$

116. *In any triangle, the square of any side is equal to the sum of the squares of the other two sides, minus twice their product into the cosine of their included angle.*

CASE I. *When the included angle A is acute.*

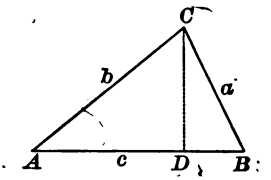


FIG. 1.

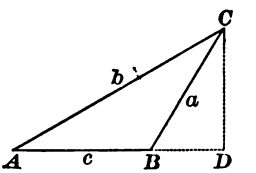


FIG. 2.

There will be two cases, according as the remaining angles are both acute (Fig. 1), or one of them obtuse (Fig. 2).

In each case let CD be drawn perpendicular to AB.

Then in Fig. 1, $BD = c - AD$,

and in Fig. 2, $BD = AD - c$.

Squaring, we have in either case,

$$\overline{BD}^2 = \overline{AD}^2 + c^2 - 2c \times AD.$$

Adding $\overline{CD}^2$ to both members,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{CD}^2 + c^2 - 2c \times AD.$$

But, $\overline{BD}^2 + \overline{CD}^2 = a^2$, and $\overline{AD}^2 + \overline{CD}^2 = b^2$.

Also, $\cos A = \frac{AD}{b}$, or $AD = b \cos A$.

Therefore, $a^2 = b^2 + c^2 - 2bc \cos A.$ (52)

Digitized by Google

CASE II. *When the included angle A is obtuse.*

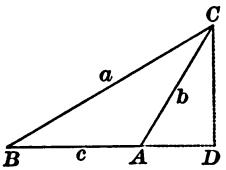


FIG. 3.

In Fig. 3, $BD = AD + c$.

Squaring, and adding $\overline{CD}^2$ to both members,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{CD}^2 + c^2 + 2c \times AD.$$

But, $\overline{BD}^2 + \overline{CD}^2 = a^2$, and $\overline{AD}^2 + \overline{CD}^2 = b^2$.

Also, $$\cos A = \cos (180° - CAD)$$
$$= -\cos CAD \text{ (Art. 47)}$$
$$= -\frac{AD}{b}.$$

Whence, $$AD = -b \cos A.$$

Therefore, $$a^2 = b^2 + c^2 - 2bc \cos A.$$

In like manner, we may prove

$$b^2 = c^2 + a^2 - 2ca \cos B, \quad (53)$$

and $$c^2 = a^2 + b^2 - 2ab \cos C. \quad (54)$$

117. *To express the cosines of the angles of a triangle in terms of the sides of the triangle.*

From (52), Art. 116,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Whence,

$$2bc \cos A = b^2 + c^2 - a^2,$$

or, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad (55)$$

Digitized by Google

In like manner, we have

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \tag{56}$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \tag{57}$$

118. *To express the sines, cosines, and tangents of the half-angles of a triangle in terms of the sides of the triangle.*

From (55), Art. 117,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Subtracting both members from unity,

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - b^2 + 2bc - c^2}{2bc}.$$

Whence by (A), Art. 75,

$$2\sin^2 \tfrac{1}{2}A = \frac{a^2 - (b-c)^2}{2bc},$$

or,

$$\sin^2 \tfrac{1}{2}A = \frac{(a - b + c)\,(a + b - c)}{4bc}.$$

Denoting $a + b + c$ by $2s$, so that s is the half-sum of the sides of the triangle, we have

$$a - b + c = (a + b + c) - 2b = 2s - 2b = 2(s - b),$$

and

$$a + b - c = (a + b + c) - 2c = 2s - 2c = 2(s - c).$$

Hence,

$$\sin^2 \tfrac{1}{2}A = \frac{4(s-b)\,(s-c)}{4bc},$$

or,

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)\,(s-c)}{bc}}. \tag{58}$$

In like manner, we may prove

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s-c)\,(s-a)}{ca}}, \tag{59}$$

and

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(s-a)\,(s-b)}{ab}}. \tag{60}$$

Digitized by Google

Again, adding both members of (55) to unity, we have

$$1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}=\frac{b^2+2bc+c^2-a^2}{2bc}.$$

Whence by (A), Art. 75,

$$2\cos^2\tfrac{1}{2}A=\frac{(b+c)^2-a^2}{2bc},$$

or,
$$\cos^2\tfrac{1}{2}A=\frac{(b+c+a)\ (b+c-a)}{4bc}.$$

But, $b+c+a=2s,$

and $b+c-a=(b+c+a)-2a=2\ (s-a).$

Hence,
$$\cos^2\tfrac{1}{2}A=\frac{4s\ (s-a)}{4bc},$$

or,
$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}. \qquad (61)$$

In like manner,

$$\cos\tfrac{1}{2}B=\sqrt{\frac{s(s-b)}{ca}}, \qquad (62)$$

and
$$\cos\tfrac{1}{2}C=\sqrt{\frac{s(s-c)}{ab}}. \qquad (63)$$

Dividing (58) by (61), we obtain

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)\ (s-c)}{bc}}\sqrt{\frac{bc}{s(s-a)}}$$

$$=\sqrt{\frac{(s-b)\ (s-c)}{s(s-a)}}. \qquad (64)$$

In like manner,

$$\tan\tfrac{1}{2}B=\sqrt{\frac{(s-c)\ (s-a)}{s(s-b)}}, \qquad (65)$$

and
$$\tan\tfrac{1}{2}C=\sqrt{\frac{(s-a)\ (s-b)}{s(s-c)}}. \qquad (66)$$

Note. Since each angle of a triangle is less than 180°, its half is less than 90°; hence the *positive sign* must be taken before the radical in each of the formulæ of Art. 118.

Digitized by Google

AREA OF AN OBLIQUE TRIANGLE.

119. CASE I. *Given two sides and their included angle.*

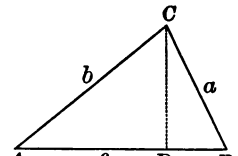


FIG. 1.

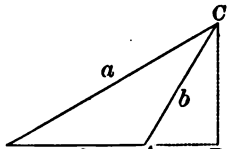


FIG. 2.

There will be two cases, according as the included angle A is acute (Fig. 1), or obtuse (Fig. 2).

In each case let CD be drawn perpendicular to AB.

Then, denoting the area of the triangle by K, we have by Geometry,

$$2K = c \times CD.$$

But in Fig. 1, $\sin A = \frac{CD}{b}$,

and in Fig. 2, $\sin A = \sin (180° - CAD)$

$$= \sin CAD \text{ (Art. 47)} = \frac{CD}{b}.$$

Whence, in either figure,

$$CD = b \sin A.$$

Therefore, $$2K = bc \sin A. \qquad (67)$$

In like manner,

$$2K = ca \sin B, \qquad (68)$$

and $$2K = ab \sin C. \qquad (69)$$

CASE II. *Given a side and all the angles.*

By (69), $$2K = ab \sin C.$$

But by Art. 113,

$$\frac{b}{a} = \frac{\sin B}{\sin A}, \text{ or } b = \frac{a \sin B}{\sin A}.$$

Digitized by Google

Substituting, $$2K = a \times \frac{a \sin B}{\sin A} \times \sin C$$

$$= \frac{a^2 \sin B \sin C}{\sin A}. \qquad (70)$$

In like manner,

$$2K = \frac{b^2 \sin C \sin A}{\sin B}, \qquad (71)$$

and $$2K = \frac{c^2 \sin A \sin B}{\sin C}. \qquad (72)$$

CASE III. *Given the three sides.*

By (67), $$2K = bc \sin A$$

$$= 2bc \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \text{ (Art. 74).}$$

Dividing by 2, and substituting the values of $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ from Art. 118, we have

$$K = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}. \qquad (73)$$

Digitized by Google

IX. SOLUTION OF OBLIQUE TRIANGLES.

120. In the solution of plane oblique triangles we may distinguish four cases:

1. *Given a side and any two angles.*
2. *Given two sides and their included angle.*
3. *Given the three sides.*
4. *Given two sides and the angle opposite to one of them.*

Case I.

121. *Given a side and any two angles.*

The third angle may be found by Geometry, and then by aid of Art. 113 the remaining sides may be calculated.

The triangle is always possible for any values of the given elements, provided the sum of the given angles is $< 180°$.

1. Given $b = 20$, $A = 104°$, $B = 19°$. Find a, c, and C.

$$C = 180° - (A + B) = 180° - 123° = 57°.$$

By Art. 113, $\frac{a}{b} = \frac{\sin A}{\sin B}$, and $\frac{c}{b} = \frac{\sin C}{\sin B}$.

That is, $a = b \sin A \csc B$, and $c = b \sin C \csc B$.

Whence, $\log a = \log b + \log \sin A + \log \csc B$,

and $\log c = \log b + \log \sin C + \log \csc B$.

$\log b = 1.3010$	$\log b = 1.3010$
$\log \sin A = 9.9869 - 10$	$\log \sin C = 9.9236 - 10$
$\log \csc B = 0.4874$	$\log \csc B = 0.4874$
$\log a = 1.7753$	$\log c = 1.7120$
$\therefore a = 59.61.$	$\therefore c = 51.52.$

Note. To find the log cosecant of an angle, subtract the log sine from $10 - 10$. To find $\log \sin 104°$, take either $\log \cos 14°$ or $\log \sin 76°$. (See pages 7 and 10 of the explanation of the tables.)

Digitized by Google

EXAMPLES.

Solve the following triangles:

2. Given $a = 10$, $A = 38°$, $B = 77° 10'$.

3. Given $b = .8037$, $B = 52° 20'$, $C = 101° 40'$.

4. Given $c = .032$, $A = 36° 8'$, $B = 44° 27'$.

5. Given $b = 29.01$, $A = 87° 40'$, $C = 33° 15'$.

6. Given $a = 5.42$, $B = 98° 22'$, $C = 41° 1'$.

7. Given $c = .0161$, $A = 35° 15'$, $C = 123° 39'$.

8. Given $a = 400$, $A = 54° 28'$, $C = 60°$.

9. Given $b = 314.29$, $A = 67° 22'$, $B = 57° 51'$.

10. Given $c = 7.86$, $B = 32° 2' 52''$, $C = 43° 25' 26''$.

Case II.

122. *Given two sides and their included angle.*

Since one angle is known, the sum of the remaining angles may be found, and then their difference may be calculated by aid of Art. 114.

Knowing the sum and difference of the angles, the angles themselves may be obtained, and then the remaining side may be computed as in Case I.

The triangle is possible for any values of the data.

1. Given $a = 167$, $c = 82$, $B = 98°$. Find A, C, and b.

By Geometry, $A + C = 180° - B = 82°$.

By Art. 114, $$\frac{a+c}{a-c} = \frac{\tan \frac{1}{2}(A+C)}{\tan \frac{1}{2}(A-C)},$$

or, $$\tan \tfrac{1}{2}(A-C) = \frac{a-c}{a+c} \tan \tfrac{1}{2}(A+C).$$

Whence, $$\log \tan \tfrac{1}{2}(A-C) = \log (a-c) + \operatorname{colog}(a+c) + \log \tan \tfrac{1}{2}(A+C).$$

Digitized by Google

$$a - c = 85 \qquad \log = 1.9294$$
$$a + c = 249 \qquad \operatorname{colog} = 7.6038$$
$$\tfrac{1}{2}(A + C) = 41^\circ \qquad \log \tan = 9.9392$$
$$\log \tan \tfrac{1}{2}(A - C) = 9.4724$$
$$\therefore \tfrac{1}{2}(A - C) = 16^\circ\, 31.7'.$$

Hence, $A = \tfrac{1}{2}(A + C) + \tfrac{1}{2}(A - C) = 57^\circ\, 31.7'$,

and $C = \tfrac{1}{2}(A + C) - \tfrac{1}{2}(A - C) = 24^\circ\, 28.3'$.

To find the remaining side, we have by Art 113,

$$b = \frac{a \sin B}{\sin A} = a \sin B \csc A.$$

Whence, $\log b = \log a + \log \sin B + \log \csc A.$

$$\log a = 2.2227$$
$$\log \sin B = 9.9958$$
$$\log \csc A = 0.0739$$
$$\log b = 2.2924$$
$$\therefore b = 196.05.$$

EXAMPLES.

Solve the following triangles:

2. Given $a = 27$, $c = 15$, $B = 46^\circ$.
3. Given $a = 486$, $b = 347$, $C = 51^\circ\, 36'$.
4. Given $b = 2.302$, $c = 3.567$, $A = 62^\circ$.
5. Given $a = .3$, $b = .363$, $C = 124^\circ\, 56'$.
6. Given $b = 1192.1$, $c = 356.3$, $A = 26^\circ\, 16'$.
7. Given $a = 7.4$, $c = 11.439$, $B = 82^\circ\, 26'$.
8. Given $a = 53.27$, $b = 41.61$, $C = 78^\circ\, 33'$.
9. Given $b = .02668$, $c = .05092$, $A = 115^\circ\, 47'$.
10. Given $a = 51.38$, $c = 67.94$, $B = 79^\circ\, 12'\, 34''$.

Digitized by Google

Case III.

123. *Given the three sides.*

The angles might be calculated by the formulæ of Art. 117; but as these are not adapted to logarithmic computation, it is more convenient to use the formulæ of Art. 118.

Each of the three angles should be computed trigonometrically, as we then have a check on the work, since their sum should be 180°.

If all the angles are to be computed, the *tangent* formulæ are the most convenient, as only four different logarithms are required. If but one angle is required, the *cosine* formulæ will be found to involve the least work.

The triangle is possible for any values of the data, provided no side is greater than the sum of the other two.

If all the angles are required, and the tangent formulæ are used, they may be conveniently modified as follows:

By Art. 118, $\tan \frac{1}{2}A = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Denoting $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ by r, we have

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}.$$

In like manner,

$$\tan \tfrac{1}{2}B = \frac{r}{s-b}, \text{ and } \tan \tfrac{1}{2}C = \frac{r}{s-c}.$$

1. Given $a = 2.5$, $b = 2.79$, $c = 2.33$; find A, B, and C.

In this case, $2s = a + b + c = 7.62$, and $s = 3.81$.

Whence, $s - a = 1.31$, $s - b = 1.02$, and $s - c = 1.48$.

By logarithms, we have

$$\log r = \tfrac{1}{2}[\log (s-a) + \log (s-b) + \log (s-c) + \operatorname{colog} s].$$

Also, $\log \tan \frac{1}{2}A = \log r - \log (s-a)$,

$\log \tan \frac{1}{2}B = \log r - \log (s-b)$,

$\log \tan \frac{1}{2}C = \log r - \log (s-c)$.

$\log (s-a) = 0.1173$ $\qquad$ $\log r = 9.8576 - 10$

$\log (s-b) = 0.0086$ $\qquad$ $\log (s-b) = 0.0086$

$\log (s-c) = 0.1703$ $\qquad$ $\log \tan \frac{1}{2}B = 9.8490 - 10$

$\operatorname{colog} s = 9.4191 - 10$ $\qquad$ $\frac{1}{2}B = 35° \ 14.1'$

$2)\overline{9.7153 - 10}$ $\qquad$ $\therefore B = 70° \ 28.2'$.

$\therefore \log r = 9.8576 - 10$

$\log r = 9.8576 - 10$ $\qquad$ $\log r = 9.8576 - 10$

$\log (s-a) = 0.1173$ $\qquad$ $\log (s-c) = 0.1703$

$\log \tan \frac{1}{2}A = 9.7403 - 10$ $\qquad$ $\log \tan \frac{1}{2}C = 9.6873 - 10$

$\frac{1}{2}A = 28° \ 48.3'$ $\qquad$ $\frac{1}{2}C = 25° \ 57.2'$

$\therefore A = 57° \ 36.6'$. $\qquad$ $\therefore C = 51° \ 54.4'$.

Check, $A + B + C = 179° \ 59.2'$.

2. Given $a = 7$, $b = 11$, $c = 9.6$; find B.

By Art. 118, $\cos \frac{1}{2}B = \sqrt{\frac{s(s-b)}{ca}}$.

Or, $\log \cos \frac{1}{2}B = \frac{1}{2}[\log s + \log (s-b) + \operatorname{colog} c + \operatorname{colog} a]$.

In this case, $2s = a + b + c = 27.6$.

Whence, $s = 13.8$, and $s - b = 2.8$.

$\log s = 1.1399$

$\log (s-b) = 0.4472$

$\operatorname{colog} c = 9.0177 - 10$

$\operatorname{colog} a = 9.1549 - 10$

$2)\overline{19.7597 - 20}$

$\log \cos \frac{1}{2}B = 9.8798 - 10$

$\frac{1}{2}B = 40° \ 41.8'$

$\therefore B = 81° \ 23.6'$.

Digitized by Google

EXAMPLES.

Solve the following triangles:

3. Given $a = 2$, $b = 3$, $c = 4$.

4. Given $a = 4$, $b = 7$, $c = 6$.

5. Given $a = 5.6$, $b = 4.3$, $c = 4.9$.

6. Given $a = .23$, $b = .26$, $c = .198$.

7. Given $a = 79.3$, $b = 94.2$, $c = 66.9$.

8. Given $a = 321$, $b = 361$, $c = 402$.

9. Given $a = .641$, $b = .529$, $c = .702$.

10. Given $a = 3.019$, $b = 6.731$, $c = 4.228$.

CASE IV.

124. *Given two sides and the angle opposite to one of them.*

It was stated in Art. 106 that a triangle is in general completely determined when three of its elements are known, provided one of them is a side. The only exceptions occur in Case IV.

To illustrate, let us consider the following:

1. Given $a = 52.1$, $b = 61.2$, $A = 31° 26'$. Required B, C, and c.

By Art. 113, $$\frac{\sin B}{\sin A} = \frac{b}{a}, \text{ or } \sin B = \frac{b \sin A}{a}.$$

Whence, $$\log \sin B = \log b + \operatorname{colog} a + \log \sin A.$$

$$\log b = 1.7868$$
$$\operatorname{colog} a = 8.2832 - 10$$
$$\log \sin A = 9.7173 - 10$$
$$\log \sin B = 9.7873 - 10$$

$\therefore B = 37° 47.5'$, from the table.

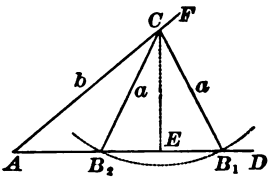
Digitized by Google

But in determining the angle corresponding, attention must be paid to the fact that an angle and its supplement have the same sine (Art. 47).

Therefore another value of B will be $180° - 37° 47.5'$, or $142° 12.5'$; and calling these values B_1 and B_2, we have

$$B_1 = 37° 47.5', \text{ and } B_2 = 142° 12.5'.$$

Note. The reason for this ambiguity is at once apparent when we attempt to construct the triangle from the data.



We first lay off the angle DAF equal to $31° 26'$, and on AF take $AC = 61.2$. With C as a centre, and a radius equal to 52.1, describe an arc cutting AD at B_1 and B_2. Then either of the triangles AB_1C or AB_2C satisfies the given conditions.

The two values of B which were obtained are the values of the angles AB_1C and AB_2C respectively; and it is evident geometrically that these angles are supplementary.

To complete the solution, denote the angles ACB_1 and ACB_2 by C_1 and C_2, and the sides AB_1 and AB_2 by c_1 and c_2.

Then, $C_1 = 180° - (A + B_1) = 180° - 69° 13.5' = 110° 46.5'$,

and $C_2 = 180° - (A + B_2) = 180° - 173° 38.5' = 6° 21.5'$.

Again, $\dfrac{c_1}{a} = \dfrac{\sin C_1}{\sin A}$, and $\dfrac{c_2}{a} = \dfrac{\sin C_2}{\sin A}$.

Whence, $c_1 = a \sin C_1 \csc A$, and $c_2 = a \sin C_2 \csc A$.

$\log a = 1.7168$	$\log a = 1.7168$
$\log \sin C_1 = 9.9708$	$\log \sin C_2 = 9.0443$
$\log \csc A = 0.2827$	$\log \csc A = 0.2827$
$\log c_1 = 1.9703$	$\log c_2 = 1.0438$
$\therefore c_1 = 93.4.$	$\therefore c_2 = 11.06.$

Digitized by Google

125. Whenever an angle of an oblique triangle is determined from its *sine*, both the acute and obtuse values must be retained as solutions, unless one of them can be shown by other considerations to be inadmissible; and hence there may sometimes be two solutions, sometimes only one, and sometimes none, in an example under Case IV.

I. Let the data be a, b, and A, and suppose $b < a$.

Since, by Geometry, B must be $< A$, only the *acute* value of B can be taken; in this case there is but *one* solution.

II. Let the data be a, b, and A, and suppose $b > a$.

Since B must be $> A$, the triangle is impossible unless A is acute.

Again, since $\frac{\sin B}{\sin A} = \frac{b}{a}$, and b is $> a$, $\sin B$ is $> \sin A$.

Hence both the acute and obtuse values of B are $> A$, and there are *two* solutions, except in the following cases:

If the data are such as to make $\log \sin B = 0$, then $\sin B = 1$ (Art. 87) and $B = 90°$, and the triangle is a right triangle; if $\log \sin B$ is *positive*, then $\sin B$ is > 1, and the triangle is impossible.

126. The results of Art. 125 may be stated as follows:

If, of the given sides, that adjacent to the given angle is the *less*, there is but *one* solution, corresponding to the *acute* value of the opposite angle.

If the side adjacent to the given angle is the *greater*, there are *two* solutions unless the log sine of the opposite angle is 0 or positive; in which cases there are *one* solution (a *right* triangle), and *no* solution, respectively.

127. We will illustrate the above points by examples:

2. Given $a = 7.42$, $b = 3.39$, $A = 105° 13'$; find B.

Since b is $< a$, there is but *one* solution, corresponding to the *acute* value of B.

Digitized by Google

We have, $\sin B = \dfrac{b \sin A}{a}$.

$$\log b = 0.5302$$
$$\operatorname{colog} a = 9.1296$$
$$\log \sin A = 9.9845$$
$$\log \sin B = 9.6443$$
$$\therefore B = 26^\circ\ 9.6'.$$

3. Given $b = 3$, $c = 2$, $C = 100^\circ$; find B.

Since b is $> c$, and C is obtuse, the triangle is impossible.

4. Given $a = 22.764$, $c = 50$, $A = 27^\circ\ 4.8'$; find C.

We have, $\sin C = \dfrac{c \sin A}{a}$.

$$\log c = 1.6990$$
$$\operatorname{colog} a = 8.6428$$
$$\log \sin A = 9.6582$$
$$\log \sin C = 0.0000$$
$$\therefore \sin C = 1, \text{ and } C = 90^\circ.$$

Here there is but one solution; a *right* triangle.

5. Given $a = .83$, $b = .715$, $B = 61^\circ\ 47'$; find A.

We have, $\sin A = \dfrac{a \sin B}{b}$.

$$\log a = 9.9191$$
$$\operatorname{colog} b = 0.1457$$
$$\log \sin B = 9.9451$$
$$\log \sin A = 0.0099$$

Since $\log \sin A$ is positive, the triangle is impossible.

EXAMPLES.

Solve the following triangles:

6. Given $a = 5.08$, $b = 3.59$, $A = 63^\circ\ 50'$.

7. Given $b = 74.8$, $c = 62.2$, $C = 27^\circ\ 18'$.

Digitized by Google

8. Given $b = .2337$, $c = .1982$, $B = 109°$.
9. Given $a = 1.07$, $c = 1.71$, $C = 31° 53'$.
10. Given $a = .1864$, $b = .17$, $B = 63° 40'$.
11. Given $a = 50$, $c = 66$, $A = 123° 11'$.
12. Given $b = 50.3$, $c = 66.8$, $C = 32° 49'$.
13. Given $a = 8.656$, $c = 10$, $A = 59° 57'$.
14. Given $b = 5.161$, $c = 6.84$, $B = 44° 3'$.
15. Given $a = 214.56$, $b = 284.79$, $B = 104° 20'$.
16. Given $b = 3069$, $c = 1223$, $C = 55° 52'$.
17. Given $a = .7097$, $c = .5112$, $A = 35° 11'$.
18. Given $a = 106.85$, $b = 166.21$, $A = 40° 0' 21''$.
19. Given $a = .3216$, $c = .2708$, $C = 52° 24'$.
20. Given $b = 811.3$, $c = 606.4$, $B = 126° 5' 20''$.

AREA OF AN OBLIQUE TRIANGLE.

128. 1. Given $a = 18.063$, $A = 96° 30'$, $B = 35°$; find K.

By Art. 119, $$2K = \frac{a^2 \sin B \sin C}{\sin A}$$
$$= a^2 \sin B \sin C \csc A.$$

Whence,
$$\log (2K) = 2 \log a + \log \sin B + \log \sin C + \log \csc A.$$

From the data,
$$C = 180° - (A + B) = 48° 30'.$$

$\log a = 1.2568$; multiply by $2 = 2.5136$
$\log \sin B = 9.7586$
$\log \sin C = 9.8745$
$\log \csc A = 0.0028$
$\log (2K) = 2.1495$
$\therefore 2K = 141.1$, and $K = 70.55$.

Digitized by Google

EXAMPLES.

Find the areas of the following triangles:

2. Given $a = 38$, $c = 61.2$, $B = 67° 56'$.
3. Given $a = 5$, $b = 7$, $c = 6$.
4. Given $b = 2.07$, $A = 70°$, $B = 36° 23'$.
5. Given $b = 116.1$, $c = 100$, $A = 118° 16'$.
6. Given $a = 79$, $b = 94$, $c = 67$.
7. Given $a = 3.123$, $A = 53° 11'$, $B = 13° 57'$.
8. Given $b = .439$, $A = 76° 38'$, $C = 40° 35'$.
9. Given $a = 23.1$, $b = 19.7$, $c = 25.2$.
10. Given $a = .3228$, $c = .9082$, $B = 60° 16'$.
11. Given $c = 80.25$, $B = 100° 5'$, $C = 31° 44'$.
12. Given $a = .010168$, $b = .018225$, $C = 11° 18' 26''$.
13. Given $a = 5.82$, $b = 6$, $c = 4.26$.

MISCELLANEOUS EXAMPLES.

129. 1. From a point in the same horizontal plane with the base of a tower, the angle of elevation of its top is $52° 39'$, and from a point 100 feet further away it is $35° 16'$. Required the height of the tower, and its distance from each of the points of observation.

2. In a field $ABCD$, the sides AB, BC, CD, and DA are 155, 236, 252, and 105 rods, respectively, and the length of a line from A to C is 311 rods. Find the area of the field.

3. From the top of a bluff, the angles of depression of two posts in the plain below, in line with the observer and 1000 feet apart, are found to be $27° 40'$ and $9° 33'$, respectively. What is the height of the bluff above the plain?

4. Two yachts start at the same time from the same point, and sail, one due north at the rate of 10.44 miles an

Digitized by Google

hour, and the other due northeast at the rate of 7.71 miles an hour. How far apart are they at the end of 40 minutes?

5. A ship is sailing due southwest at the rate of 8 miles an hour. At 10.30 A.M., a lighthouse is observed to bear 30° west of north, and at 12.15 P.M., it is observed to bear 15° east of north. Find the distance of the lighthouse from each position of the ship.

6. Wishing to find the distance of an inaccessible object A from a position B, I measure a line BC, 208.3 feet in length. The angles ABC and ACB are measured, and found to be 126° 35′ and 31° 48′, respectively. Required the distance AB.

7. A flagpole 40 feet in height stands on the top of a tower. From a position near the base of the tower, the angles of elevation of the top and bottom of the pole are 38° 53′ and 20° 18′, respectively. Required the distance and height of the tower.

8. A surveyor observes that his position A is exactly in line with two inaccessible objects B and C. He measures a line AD, 500 feet in length, making the angle $BAD = 60°$, and at D observes the angles ADB and BDC to be 40° and 60°, respectively. Required the distance BC.

9. To find the distance between two buoys A and B, I measure a base line CD on the shore, 150 feet in length. At the point C the angles ACD and BCD are measured and found to be 95° and 70°, respectively; and at D the angles BDC and ADC are found to be 83° and 30°. What is the distance between the buoys?

10. The sides of a field $ABCD$ are $AB = 37$, $BC = 63$, and $DA = 20$, and the diagonals AC and BD are 75 and 42, respectively. Required the area of the field.

Digitized by Google

PART II.

SPHERICAL TRIGONOMETRY.

X. GEOMETRICAL DEFINITIONS AND PRINCIPLES.

130. If a triedral angle is formed with its vertex at the centre of a sphere, it intercepts on the surface a *spherical triangle.*

131. The triangle is bounded by three arcs of great circles called its sides, which measure the face angles of the triedral angle.

The *angles* of the spherical triangle are the diedral angles of the triedral angle; and by Geometry, each is measured by the angle between two straight lines drawn, one in each face, and perpendicular to the edge at the same point.

132. The sides of a spherical triangle, being arcs, are usually expressed in degrees.

If the length of a side in terms of some *linear* unit is desired, it may be obtained by finding the ratio of its arc to 360°, and multiplying the result by the length of the circumference of a great circle.

133. *Spherical Trigonometry* treats of the trigonometrical relations between the elements of a spherical triangle; or what is the same thing, between the face and diedral angles of the triedral angle which intercepts it.

Digitized by Google

134. The face and diedral angles are not altered in magnitude by varying the radius of the sphere, and hence the relations between the sides and angles of a spherical triangle are independent of the length of the radius.

135. We shall limit ourselves in this work to such triangles as are considered in Geometry, where each angle is less than two right angles, and each side less than the semicircumference of a great circle; that is, where each element is less than 180°.

136. The proofs of the following properties of spherical triangles may be found in any treatise on Solid Geometry:

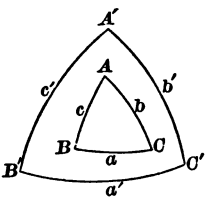
(*a*) Either side of a spherical triangle is less than the sum of the other two sides.

(*b*) If two sides of a spherical triangle are unequal, the angles opposite them are unequal, and the greater angle lies opposite the greater side; and conversely.

(*c*) The sum of the sides of a spherical triangle is less than 360°.

(*d*) The sum of the angles of a spherical triangle is greater than 180°, and less than 540°.

(*e*) If $A'B'C'$ is the polar triangle of ABC, *i.e.*, if A, B, and C are the poles of the arcs a', b', and c', respectively, then conversely, ABC is the polar triangle of $A'B'C'$.



(*f*) In two polar triangles, each angle of one is measured by the supplement of the side lying opposite to it in the other.

Digitized by Google

That is,

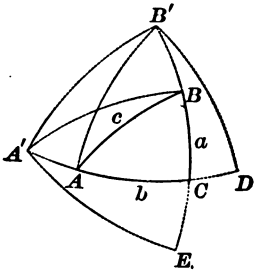
$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c',$$
$$A' = 180° - a, \quad B' = 180° - b, \quad C' = 180° - c.$$

137. A spherical triangle is called *tri-rectangular* when it has three right angles; each of its sides is a quadrant, and each vertex is the pole of the opposite side.

138. I. Let C be the right angle of the spherical right triangle ABC, and suppose $a < 90°$ and $b < 90°$.

Complete the tri-rectangular triangle $A'B'C$.

Also, since B' is the pole of AC, and A' of BC, construct the tri-rectangular triangles $AB'D$ and $A'BE$.

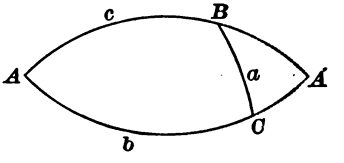


Then since A and B lie on the same side of $B'D$, AB or c is $< 90°$.

Since BC is $< B'C$, the angle A is $< B'AD$, or $< 90°$.

Since AC is $< A'C$, the angle B is $< A'BE$, or $< 90°$.

II. Suppose $a < 90°$ and $b > 90°$.



Complete the lune $ABA'C$.

Digitized by Google

Then in the right triangle $A'BC$, $A'C = 180° - b$.

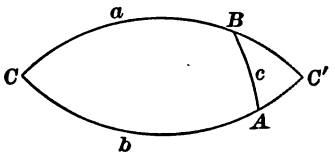
That is, the sides a and $A'C$ of the triangle $A'BC$ are each $< 90°$; and by I., $A'B$ and the angles A' and $A'BC$ are each $< 90°$.

But, $c = 180° - A'B$, $A = A'$, and $B = 180° - A'BC$.

Whence, c is $> 90°$, $A < 90°$, and $B > 90°$.

In like manner, if a is $> 90°$ and $b < 90°$, then c is $> 90°$, $A > 90°$, and $B < 90°$.

III. Suppose $a > 90°$ and $b > 90°$.



Complete the lune $ACBC'$.

Then in the right triangle ABC', $AC' = 180° - b$, and $BC' = 180° - a$.

That is, the sides AC' and BC' of the triangle ABC' are each $< 90°$; and by I., AB and the angles BAC' and ABC' are each $< 90°$.

But, $A = 180° - BAC'$, and $B = 180° - ABC'$.

Whence, c is $< 90°$, $A > 90°$, and $B > 90°$.

Hence, in any spherical right triangle:

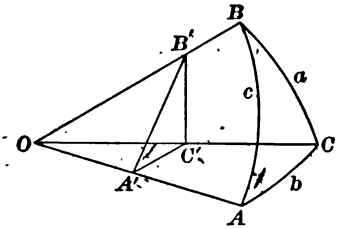
1. *If the sides including the right angle are in the same quadrant, the hypotenuse is* $< 90°$; *if they are in different quadrants, the hypotenuse is* $> 90°$.

2. *An angle is in the same quadrant as its opposite side.*

Digitized by Google

XI. SPHERICAL RIGHT TRIANGLES.

139. Let C be the right angle of the spherical right triangle ABC, and O the centre of the sphere.



Join OA, OB, and OC.

At any point A' of OA draw $A'B'$ and $A'C'$ perpendicular to OA, and join $B'C'$.

Then by Art. 131, the sides a, b, and c measure the angles BOC, COA, and AOB, respectively, and the angle $B'A'C'$ is equal to the angle A of the spherical triangle.

Since OA is perpendicular to $A'B'$ and $A'C'$, it is perpendicular to the plane $A'B'C'$.

Whence, since each of the planes $A'B'C'$ and OBC is perpendicular to the plane OAC, their intersection $B'C'$ is perpendicular to OAC.

Therefore $B'C'$ is perpendicular to $A'C'$ and OC'.

In the right triangle $OA'B'$, we have

$$\cos c = \cos A'OB' = \frac{OA'}{OB'} = \frac{OC'}{OB'} \times \frac{OA'}{OC'}.$$

But in the right triangles $OB'C'$ and $OC'A'$,

$$\frac{OC'}{OB'} = \cos a, \text{ and } \frac{OA'}{OC'} = \cos b.$$

Whence, $\cos c = \cos a \cos b.$ **(74)**

Digitized by Google

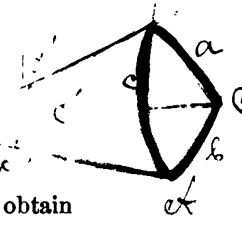
$$\text{Again, } \sin A = \sin B'A'C' = \frac{B'C'}{A'B'} = \frac{\frac{B'C'}{OB'}}{\frac{A'B'}{OB'}} = \frac{\sin a}{\sin c}, \quad (75)$$

$$\text{and } \cos A = \cos B'A'C' = \frac{A'C'}{A'B'} = \frac{\frac{A'C'}{OA'}}{\frac{A'B'}{OA'}} = \frac{\tan b}{\tan c}. \quad (76)$$

In like manner we have,

$$\sin B = \frac{\sin b}{\sin c}, \quad (77)$$

$$\text{and } \cos B = \frac{\tan a}{\tan c}. \quad (78)$$



140. From (75) and (76), we obtain

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin a}{\sin c} \times \frac{\tan c}{\tan b} = \frac{\sin a}{\cos c \tan b}.$$

Whence by (74),

$$\tan A = \frac{\sin a}{\cos a \cos b \tan b} = \frac{\tan a}{\sin b}. \quad (79)$$

Similarly,

$$\tan B = \frac{\tan b}{\sin a}. \quad (80)$$

141. Since $\sin a = \cos a \tan a$ (Art. 20), (75) may be written

$$\sin A = \frac{\cos a \tan a}{\cos c \tan c} = \frac{\frac{\tan a}{\tan c}}{\frac{\cos c}{\cos a}}.$$

Whence by (74) and (78),

$$\sin A = \frac{\cos B}{\cos b}. \quad (81)$$

Similarly,

$$\sin B = \frac{\cos A}{\cos a}. \quad (82)$$

Digitized by Google

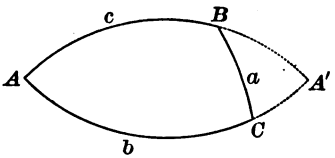
142. From (**74**), (**81**), and (**82**), we have

$$\begin{aligned}\cos c &= \cos a \cos b \\ &= \frac{\cos A}{\sin B} \times \frac{\cos B}{\sin A} \\ &= \cot A \cot B. \end{aligned} \qquad (83)$$

143. The proofs of Art. 139 cannot be regarded as general, for in the construction of the figure we have assumed a and b, and therefore c and A (Art. 138), to be less than 90°.

To prove formulæ (**74**) to (**78**) universally, it is necessary to consider two additional cases:

CASE I. *When one of the sides a and b is $< 90°$, and the other $> 90°$.*



In the right triangle ABC, let a be $< 90°$ and $b > 90°$.

Complete the lune $ABA'C$; then in the triangle $A'BC$,

$$A'B = 180° - c, \qquad A' = A,$$
$$A'C = 180° - b, \qquad A'BC = 180° - B.$$

But by Art. 138, c is $> 90°$, $A < 90°$, and $B > 90°$.

Hence each element, except the right angle, of the right triangle $A'BC$ is $< 90°$, and we have by Art. 139,

$$\cos A'B = \cos a \cos A'C,$$

$$\sin A' = \frac{\sin a}{\sin A'B}, \qquad \sin A'BC = \frac{\sin A'C}{\sin A'B},$$

$$\cos A' = \frac{\tan A'C}{\tan A'B}, \qquad \cos A'BC = \frac{\tan a}{\tan A'B}.$$

Digitized by Google

Putting for $A'B$, $A'C$, A', and $A'BC$ their values, we have

$$\cos(180° - c) = \cos a \cos(180° - b),$$

$$\sin A = \frac{\sin a}{\sin(180° - c)}, \quad \sin(180° - B) = \frac{\sin(180° - b)}{\sin(180° - c)},$$

$$\cos A = \frac{\tan(180° - b)}{\tan(180° - c)}, \quad \cos(180° - B) = \frac{\tan a}{\tan(180° - c)}.$$

Whence by Art. 47,

$$-\cos c = \cos a\,(-\cos b),$$

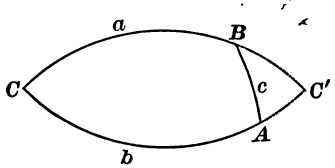
$$\sin A = \frac{\sin a}{\sin c}, \qquad \sin B = \frac{\sin b}{\sin c},$$

$$\cos A = \frac{-\tan b}{-\tan c}, \qquad -\cos B = \frac{\tan a}{-\tan c};$$

and we obtain the formulæ (**74**) to (**78**) as before.

In like manner, the formulæ may be proved in the case where a is $> 90°$ and $b < 90°$.

CASE II. *When both a and b are $> 90°$.*



In the right triangle ABC, let a and b be $> 90°$.

Complete the lune $ACBC'$.

By Art. 138, c is $< 90°$, $A > 90°$, and $B > 90°$.

Hence each element, except the right angle, of the right triangle ABC' is $< 90°$, and we have by Art. 139,

$$\cos c = \cos AC' \cos BC',$$

$$\sin BAC' = \frac{\sin BC'}{\sin c}, \qquad \sin ABC' = \frac{\sin AC'}{\sin c},$$

$$\cos BAC' = \frac{\tan AC'}{\tan c}, \qquad \cos ABC' = \frac{\tan BC'}{\tan c}.$$

Digitized by Google

Putting for AC', BC', BAC', and ABC' their values, we have

$$\cos c = \cos(180° - a)\cos(180° - b),$$

$$\sin(180° - A) = \frac{\sin(180° - a)}{\sin c}, \quad \sin(180° - B) = \frac{\sin(180° - b)}{\sin c},$$

$$\cos(180° - A) = \frac{\tan(180° - b)}{\tan c}, \quad \cos(180° - B) = \frac{\tan(180° - a)}{\tan c}.$$

Whence by Art. 47,

$$\cos c = (-\cos a)(-\cos b),$$

$$\sin A = \frac{\sin a}{\sin c}, \qquad \sin B = \frac{\sin b}{\sin c},$$

$$-\cos A = \frac{-\tan b}{\tan c}, \qquad -\cos B = \frac{-\tan a}{\tan c};$$

and we obtain the formulæ (**74**) to (**78**) as before.

144. The formulæ of Arts. 139 to 142 are collected below for the convenience of the student:

$$\cos c = \cos a \cos b.$$

$$\sin A = \frac{\sin a}{\sin c}. \qquad \sin B = \frac{\sin b}{\sin c}.$$

$$\cos A = \frac{\tan b}{\tan c}. \qquad \cos B = \frac{\tan a}{\tan c}.$$

$$\tan A = \frac{\tan a}{\sin b}. \qquad \tan B = \frac{\tan b}{\sin a}.$$

$$\sin A = \frac{\cos B}{\cos b}. \qquad \sin B = \frac{\cos A}{\cos a}.$$

$$\cos c = \cot A \cot B.$$

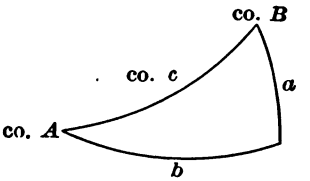
By comparing the formulæ for the sines, cosines, and tangents of A and B with the corresponding forms for plane triangles as given in Arts. 10 and 14, no difficulty will be found in retaining them in the memory.

Digitized by Google

NAPIER'S RULES OF CIRCULAR PARTS.

145. These are two artificial rules which include all the formulæ of the preceding article.

In any spherical right triangle, the elements a and b, and the complements of the elements A, B, and c (written in abbreviated form, co. A, co. B, and co. c), are called the *circular parts*.



If we suppose them arranged in the order in which the letters occur in the triangle, any one of the five may be taken and called the *middle part;* the two immediately adjacent are called the *adjacent parts*, and the remaining two the *opposite parts.*

Then Napier's rules are:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

146. Napier's rules may be proved by taking each of the circular parts in succession as the middle part, and showing that the results agree with the formulæ of Art. 144.

1. If a is the middle part, b and co. B are the adjacent parts, and co. c and co. A the opposite parts. Then the rules give

$$\sin a = \tan b \tan(\text{co. } B),$$

and $$\sin a = \cos(\text{co. } c)\cos(\text{co. } A);$$

or, by Art. 14, $\sin a = \tan b \cot B$, and $\sin a = \sin c \sin A$;

which agree with (**80**) and (**75**).

Digitized by Google

2. If b is the middle part, a and co. A are the adjacent parts, and co. c and co. B the opposite parts. Then,

$$\sin b = \tan a \tan (\text{co. } A)$$
$$= \tan a \cot A,$$

and
$$\sin b = \cos (\text{co. } c) \cos (\text{co. } B)$$
$$= \sin c \sin B;$$

which agree with (**79**) and (**77**).

3. If co. c is the middle part, co. A and co. B are the adjacent parts, and a and b the opposite parts. Then,

$$\sin (\text{co. } c) = \tan (\text{co. } A) \tan (\text{co. } B),$$

or
$$\cos c = \cot A \cot B;$$

and
$$\sin (\text{co. } c) = \cos a \cos b,$$

or
$$\cos c = \cos a \cos b;$$

which agree with (**83**) and (**74**).

4. If co. A is the middle part, b and co. c are the adjacent parts, and a and co. B the opposite parts. Then,

$$\sin (\text{co. } A) = \tan b \tan (\text{co. } c), \text{ or } \cos A = \tan b \cot c,$$

and $\sin (\text{co. } A) = \cos a \cos (\text{co. } B)$, or $\cos A = \cos a \sin B$;

which agree with (**76**) and (**82**).

5. If co. B is the middle part, a and co. c are the adjacent parts, and b and co. A the opposite parts. Then,

$$\sin (\text{co. } B) = \tan a \tan (\text{co. } c), \text{ or } \cos B = \tan a \cot c,$$

and $\sin (\text{co. } B) = \cos b \cos (\text{co. } A)$, or $\cos B = \cos b \sin A$;

which agree with (**78**) and (**81**).

147. Writers on Trigonometry differ as to the practical value of Napier's rules; but in the opinion of the highest authorities, it seems to be regarded as preferable to attempt to remember the formulæ by comparing them with the analogous forms for plane triangles, as stated in Art. 144.

Digitized by Google

SOLUTION OF SPHERICAL RIGHT TRIANGLES.

148. To solve a spherical right triangle, two elements must be given in addition to the right angle.

There may be six cases:

1. *Given the hypotenuse and an adjacent angle.*
2. *Given an angle and its opposite side.*
3. *Given an angle and its adjacent side.*
4. *Given the hypotenuse and another side.*
5. *Given the two sides a and b.*
6. *Given the two angles A and B.*

149. Either of the above may be solved by aid of Art. 144.

The formula for computing either of the remaining elements when any two are given may be found by the following rule:

Take that formula which involves the given parts and the required part.

If *all* the remaining elements are required, the following rule may be found convenient in selecting the formulæ:

Take the three formulæ which involve the given parts.

150. It is convenient in the solution to have a check on the logarithmic work, which may be done in every case without the necessity of looking out any new logarithms.

Examples of this will be found in Art. 153.

The check formula for any particular case may be selected from the set in Art. 144 by the following rule:

Take that formula which involves the three required parts.

Note. If Napier's rules are used, the following rule will indicate which of the circular parts corresponding to the given elements and any required element is to be regarded as the middle part:

If these three circular parts are adjacent, take the middle one as the middle part, and the others are then adjacent parts.

If they are not adjacent, take the part which is not adjacent to either of the others as the middle part, and the others are then the opposite parts.

For the check formula, proceed as above with the circular parts corresponding to the three required elements.

Digitized by Google

Thus if c and A are the given elements,

1. To find a, consider the circular parts a, co.c, and co.A; of these, a is the middle part, and co.c and co.A are opposite parts. Then, by Napier's rules,

$$\sin a = \cos(\text{co}.c)\cos(\text{co}.A) = \sin c \sin A.$$

2. To find b, the circular parts are b, co.c, and co.A; in this case co.A is the middle part, and b and co.c are adjacent parts. Then,

$$\sin(\text{co}.A) = \tan b \tan(\text{co}.c), \text{ or } \cos A = \tan b \cot c.$$

3. To find B, the circular parts are co.B, co.c, and co.A; co.c is the middle part, and co.A and co.B are adjacent parts. Then,

$$\sin(\text{co}.c) = \tan(\text{co}.A)\tan(\text{co}.B), \text{ or } \cos c = \cot A \cot B.$$

4. For the check formula, the circular parts are a, b, and co.B; a is the middle part, and b and co.B are adjacent parts. Then,

$$\sin a = \tan b \tan(\text{co}.B) = \tan b \cot B.$$

151. In solving spherical triangles, careful attention must be paid to the *algebraic signs* of the functions; the cosines, tangents, and cotangents of angles greater than 90° being taken *negative*.

It is convenient to place the sign of each function just above or below it, as illustrated in the examples of Art. 153; the sign of the function in the first member being then determined in accordance with the principle that like signs produce +, and unlike signs produce −.

Note. In the examples after the first of Art. 153, the signs are omitted in every case where both factors of the second member are +.

152. In finding the angles corresponding, if the function is a cosine, tangent, or cotangent, its sign determines whether the angle is less or greater than 90°; that is, if it is +, the angle is $< 90°$; and if it is −, the angle is $> 90°$, and the *supplement* of the acute angle obtained from the tables must be taken (Art. 47).

If the function is a sine, since the sine of an angle is equal to the sine of its supplement, both the acute angle obtained from the tables and its supplement must be retained as solutions, unless the ambiguity can be removed by the principles of Art. 138.

Digitized by Google

EXAMPLES.

153. 1. Given $B = 33° 50'$, $a = 108°$; find A, b, and c.

By the rule of Art. 149, the formulæ from Art. 144 are,

$$\sin B = \frac{\cos A}{\cos a}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \cos B = \frac{\tan a}{\tan c}.$$

That is,

$$\underset{-}{\cos A} = \underset{-}{\cos a} \underset{+}{\sin B}, \quad \underset{+}{\tan b} = \underset{+}{\sin a} \underset{+}{\tan B}, \quad \underset{-}{\tan c} = \frac{\overset{-}{\tan a}}{\underset{+}{\cos B}}.$$

Hence,
$$\log \cos A = \log \cos a + \log \sin B,$$
$$\log \tan b = \log \sin a + \log \tan B,$$
$$\log \tan c = \log \tan a - \log \cos B.$$

Since $\cos A$ and $\tan c$ are negative, the *supplements* of the angles obtained from the tables must be taken (Art. 152).

Note. When the supplement of the angle obtained from the tables is to be taken, it is convenient to write 180° minus the element in the first member, as shown below in the cases of A and c.

By the rule of Art. 150, the check formula for this case is

$$\cos A = \frac{\tan b}{\tan c}, \text{ or } \log \cos A = \log \tan b - \log \tan c.$$

The values of $\log \tan b$ and $\log \tan c$ may be taken from the first part of the work, and their difference should be equal to the result previously found for $\log \cos A$.

$\log \cos a = 9.4900 - 10$	$\log \tan a = 0.4882$
$\log \sin B = 9.7457 - 10$	$\log \cos B = 9.9194 - 10$
$\log \cos A = 9.2357 - 10$	$\log \tan c = 0.5688$
$180° - A = 80° 5.5'$	$180° - c = 74° 53.8'$
$\therefore A = 99° 54.5'.$	$\therefore c = 105° 6.2'.$
$\log \sin a = 9.9782 - 10$	*Check.*
$\log \tan B = 9.8263 - 10$	$\log \tan b = 9.8045 - 10$
$\log \tan b = 9.8045 - 10$	$\log \tan c = 0.5688$
$\therefore b = 32° 31.1'.$	$\log \cos A = 9.2357 - 10$

Digitized by Google

2. Given $c = 70° 30'$, $A = 100°$; find a, b, and B.

By Art. 149, the three formulæ are,

$$\sin A = \frac{\sin a}{\sin c}, \quad \cos A = \frac{\tan b}{\tan c}, \quad \cos c = \cot A \cot B.$$

That is,

$$\sin a = \sin c \sin A, \quad \tan b = \tan c \cos A, \quad \cot B = \cos c \tan A.$$

The side a is determined from its sine; but the ambiguity is removed by the principles of Art. 138; for a and A must be in the same quadrant. Therefore a is $> 90°$, and the supplement of the angle obtained from the table must be taken.

By Art. 150, the check formula is

$$\tan B = \frac{\tan b}{\sin a}, \quad \text{or} \quad \sin a = \tan b \cot B.$$

Note 1. The check formula should always be expressed in terms of the functions used in determining the required parts; thus, in the case above, the check formula is transformed so as to involve $\cot B$ instead of $\tan B$.

$\log \sin c = 9.9743$	$\log \cos c = 9.5235$
$\log \sin A = 9.9934$	$\log \tan A = 0.7537$
$\log \sin a = 9.9677$	$\log \cot B = 0.2772$
$180° - a = 68° 10'$	$180° - B = 27° 50.6'$
$\therefore a = 111° 50'.$	$\therefore B = 152° 9.4'.$

$\log \tan c = 0.4509$	
$\log \cos A = 9.2397$	*Check.*
$\log \tan b = 9.6906$	$\log \tan b = 9.6906$
$180° - b = 26° 7.5'$	$\log \cot B = 0.2772$
$\therefore b = 153° 52.5'.$	$\log \sin a = 9.9678$

Note 2. We observe here a difference of .0001 in the two values of $\log \sin a$. This does not necessarily indicate an error in the work, for such a small difference might easily be due to the fact that the logarithms are only *approximately* correct to the fourth decimal place.

Digitized by Google

3. Given $a = 132° 6'$, $b = 77° 51'$; find A, B, and c.

In this case the formulæ are,

$$\tan A = \frac{\tan a}{\sin b}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \cos c = \cos a \cos b.$$

The check formula is

$$\cos c = \cot A \cot B, \quad \text{or} \quad \cos c \tan A \tan B = 1.$$

That is, $\log \cos c + \log \tan A + \log \tan B = \log 1 = 0$

$\log \tan a = 0.0440$	$\log \cos a = 9.8263$
$\log \sin b = 9.9901$	$\log \cos b = 9.3232$
$\log \tan A = 0.0539$	$\log \cos c = 9.1495$
$180° - A = 48° 32.8'$	$180° - c = 81° 53.4'$
$\therefore A = 131° 27.2'.$	$\therefore c = 98° 6.6'.$

$\log \tan b = 0.6670$	*Check.*
$\log \sin a = 9.8704$	$\log \cos c = 9.1495$
$\log \tan B = 0.7966$	$\log \tan A = 0.0539$
$\therefore B = 80° 55.4'.$	$\log \tan B = 0.7966$
	$\log 1 = 0.0000$

4. Given $A = 105° 59'$, $a = 128° 33'$; find b, B, and c.

The formulæ are,

$$\sin b = \frac{\tan a}{\tan A}, \quad \sin B = \frac{\cos A}{\cos a}, \quad \sin c = \frac{\sin a}{\sin A}.$$

The check formula is

$$\sin B = \frac{\sin b}{\sin c}.$$

In this example, each of the required parts is determined from its sine; and as the ambiguity cannot be removed by Art. 138, both the acute angle obtained from the tables and its supplement must be retained in each case.

Digitized by Google

$\log \tan a = 0.0986$ $\log \sin a = 9.8932$

$\log \tan A = 0.5430$ $\log \sin A = 9.9828$

$\log \sin b = 9.5556$ $\log \sin c = 9.9104$

$\therefore b = 21° 3.9'$, $\therefore c = 54° 26.7'$,

or $158° 56.1'$. or $125° 33.3'$.

$\log \cos A = 9.4399$ *Check.*

$\log \cos a = 9.7946$ $\log \sin b = 9.5556$

$\log \sin B = 9.6453$ $\log \sin c = 9.9104$

$\therefore B = 26° 13.5'$, $\log \sin B = 9.6452$

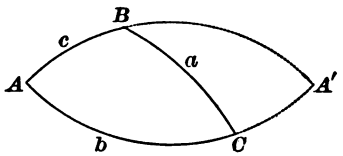
or $153° 46.5'$.

It does not follow, however, that these values can be combined promiscuously; for by Art. 138, since a is $> 90°$, with the value of b less than $90°$ must be taken the value of c greater than $90°$, and the value of B less than $90°$; while with the value of b greater than $90°$ must be taken the value of c less than $90°$, and the value of B greater than $90°$.

Thus the only solutions of the example are:

1. $b = 21° 3.9'$, $c = 125° 33.3'$, $B = 26° 13.5'$.
2. $b = 158° 56.1'$, $c = 54° 26.7'$, $B = 153° 46.5'$.

Note. The figure shows geometrically why there are two solutions in this case.



For if AB and AC are produced to A', forming the lune $ABA'C$, the triangle $A'BC$ has the side a and the angle A' equal, respectively, to the side a and the angle A of the triangle ABC, and both triangles are right-angled at C.

It is evident that the sides $A'B$ and $A'C$ and the angle $A'BC$ are the supplements of the sides c and b and the angle ABC, respectively.

Digitized by Google

Solve the following spherical right triangles:

5. Given $a = 159°$, $c = 137° 20'$.

6. Given $A = 50° 20'$, $B = 122° 40'$.

7. Given $a = 160°$, $b = 38° 30'$.

8. Given $B = 80°$, $b = 67° 40'$.

9. Given $B = 112°$, $c = 81° 50'$.

10. Given $a = 61°$, $B = 123° 40'$.

11. Given $a = 61° 40'$, $b = 144° 10'$.

12. Given $A = 99° 50'$, $a = 112°$.

13. Given $b = 15°$, $c = 152° 20'$.

14. Given $A = 62° 59'$, $B = 37° 4'$.

15. Given $A = 73° 7'$, $c = 114° 32'$.

16. Given $B = 144° 54'$, $b = 146° 32'$.

17. Given $B = 68° 18'$, $c = 47° 34'$.

18. Given $A = 161° 52'$, $b = 131° 8'$.

19. Given $a = 113° 25'$, $b = 110° 47'$.

20. Given $a = 137° 9'$, $B = 74° 51'$.

21. Given $A = 144° 54'$, $B = 101° 14'$.

22. Given $a = 69° 18'$, $c = 84° 27'$.

SOLUTION OF QUADRANTAL TRIANGLES.

154. A spherical triangle is called *quadrantal* when it has one side equal to a quadrant.

By Art. 136, (f), the polar triangle of a quadrantal triangle is a *right* triangle.

Therefore to solve a quadrantal triangle we have only to solve its polar triangle, and take the *supplements* of the parts obtained by the calculation.

1. Given $c = 90°$, $a = 67° 38'$, $b = 48° 50'$; find A, B, and C.

Digitized by Google

Denoting the polar triangle by $A'B'C'$, we have by Art. 136, (f):

$C' = 90°$, $A' = 112° 22'$, $B' = 131° 10'$; to find a', b', and c'.

By Art. 144, the formulæ for the solution are

$$\cos a' = \frac{\cos A'}{\sin B'}, \quad \cos b' = \frac{\cos B'}{\sin A'}, \quad \cos c' = \cot A' \cot B'.$$

The check formula is $\cos c' = \cos a' \cos b'$.

$\log \cos A' = 9.5804$	$\log \cot A' = 9.6143$
$\log \sin B' = 9.8767$	$\log \cot B' = 9.9417$
$\log \cos a' = 9.7037$	$\log \cos c' = 9.5560$
$\therefore 180° - a' = 59° 38.2'.$	$\therefore c' = 68° 54.8'.$

$\log \cos B' = 9.8184$	*Check.*
$\log \sin A' = 9.9660$	$\log \cos a' = 9.7037$
$\log \cos b' = 9.8524$	$\log \cos b' = 9.8524$
$\therefore 180° - b' = 44° 36.7'.$	$\log \cos c' = 9.5561$

Then in the given quadrantal triangle, we have

$$A = 180° - a' = 59° 38.2',$$
$$B = 180° - b' = 44° 36.7',$$
$$C = 180° - c' = 111° 5.2'.$$

EXAMPLES.

Solve the following quadrantal triangles:

2. Given $A = 139°$, $b = 143°$.

3. Given $A = 45° 30'$, $B = 139° 20'$.

4. Given $a = 30° 20'$, $C = 42° 40'$.

5. Given $B = 70° 12'$, $C = 106° 25'$.

6. Given $A = 105° 53'$, $a = 104° 54'$.

Digitized by Google

XII. SPHERICAL OBLIQUE TRIANGLES.

GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

155. *In any spherical triangle, the sines of the sides are proportional to the sines of their opposite angles.*

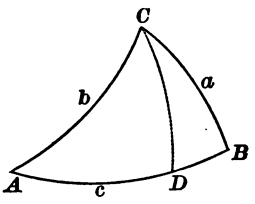


FIG. 1.

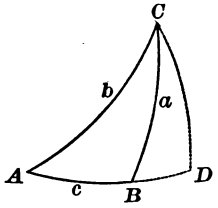


FIG. 2.

Let ABC be any spherical triangle, and draw the arc CD perpendicular to AB.

There will be two cases according as CD falls upon AB (Fig. 1), or upon AB produced (Fig. 2).

In the right triangle ACD, in either figure, we have by Art. 144,

$$\sin A = \frac{\sin CD}{\sin b}.$$

Also, in Fig. 1,

$$\sin B = \frac{\sin CD}{\sin a},$$

and in Fig. 2,
$$\begin{aligned} \sin B &= \sin (180° - CBD) \\ &= \sin CBD \quad \text{(Art. 47)} \\ &= \frac{\sin CD}{\sin a}. \end{aligned}$$

Dividing these equations, we have in either case

$$\frac{\sin A}{\sin B} = \frac{\dfrac{\sin CD}{\sin b}}{\dfrac{\sin CD}{\sin a}} = \frac{\sin a}{\sin b}. \tag{84}$$

Digitized by Google

In like manner we have,

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}, \tag{85}$$

and

$$\frac{\sin C}{\sin A} = \frac{\sin c}{\sin a}. \tag{86}$$

The above results may be expressed more compactly as follows:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

156. *In any spherical triangle, the cosine of either side is equal to the product of the cosines of the other two sides, plus the continued product of their sines and the cosine of their included angle.*

In the right triangle BCD, in Fig. 1 of the preceding article, we have by Art. 144,

$$\cos a = \cos BD \cos CD$$
$$= \cos (c - AD) \cos CD.$$

And in Fig. 2,

$$\cos a = \cos BD \cos CD$$
$$= \cos (AD - c) \cos CD.$$

Whence, in either case,

$$\cos a = \cos c \cos AD \cos CD + \sin c \sin AD \cos CD.$$

But in the right triangle ACD, by Art. 144,

$$\cos AD \cos CD = \cos b.$$

Also,

$$\sin AD \cos CD = \sin AD \frac{\cos b}{\cos AD} = \cos b \tan AD$$
$$= \sin b \frac{\tan AD}{\tan b} = \sin b \cos A \text{ (Art. 144)}.$$

Whence,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{87}$$

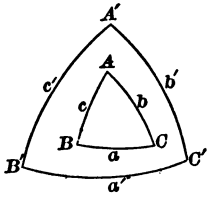
Digitized by Google

In like manner we have,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \qquad (88)$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos C. \qquad (89)$$

157. Let ABC and $A'B'C'$ be a pair of polar triangles.



Applying the theorem of Art. 156 to the side a' of the triangle $A'B'C'$, we obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Putting for a', b', c', and A' their values as given in Art. 136, (f), we have

$$\cos(180° - A) = \cos(180° - B) \cos(180° - C) + \sin(180° - B) \sin(180° - C) \cos(180° - a).$$

Whence by Art. 47,

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C (-\cos a).$$

That is,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \qquad (90)$$

In like manner,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \qquad (91)$$

and $$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \qquad (92)$$

The above proof illustrates a very important application of the theory of polar triangles in Spherical Trigonometry.

If any relation has been found between the elements of a triangle, an analogous relation may be at once derived from it, in which each side or angle is replaced by the opposite angle or side, with suitable modifications in the algebraic signs.

Digitized by Google

158. *To express the sines, cosines, and tangents of the half-angles of a spherical triangle in terms of the sides of the triangle.*

From (87), Art. 156, we obtain

$$\sin b \sin c \cos A = \cos a - \cos b \cos c,$$

or,
$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \qquad \text{(B)}$$

Subtracting both members from unity,

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}.$$

Whence by (A), Art. 75,

$$2 \sin^2 \tfrac{1}{2} A = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

But by Art. 72,

$$\cos y - \cos x = 2 \sin \tfrac{1}{2} (x + y) \sin \tfrac{1}{2} (x - y).$$

Whence,

$$2 \sin^2 \tfrac{1}{2} A = \frac{2 \sin \tfrac{1}{2} [a + (b - c)] \sin \tfrac{1}{2} [a - (b - c)]}{\sin b \sin c},$$

or,
$$\sin^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} (a + b - c) \sin \tfrac{1}{2} (a - b + c)}{\sin b \sin c}.$$

Denoting $a + b + c$ by $2s$, so that s is the half-sum of the sides of the triangle, we have

$$a + b - c = (a + b + c) - 2c = 2s - 2c = 2(s - c),$$

and
$$a - b + c = (a + b + c) - 2b = 2s - 2b = 2(s - b).$$

Whence,

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c},$$

and
$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \qquad (93)$$

Digitized by Google

In like manner we have,

$$\sin \tfrac{1}{2}B = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}}, \qquad (94)$$

and

$$\sin \tfrac{1}{2}C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}. \qquad (95)$$

Again, adding both members of (B) to unity, we have

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c},$$

or,

$$2\cos^2 \tfrac{1}{2}A = \frac{\cos a - \cos(b+c)}{\sin b \sin c}.$$

Whence, as in the proof of (93),

$$\cos^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(b+c+a)\sin \frac{1}{2}(b+c-a)}{\sin b \sin c}.$$

But, $b+c+a = 2s$, and $b+c-a = 2(s-a)$.

Whence,

$$\cos^2 \tfrac{1}{2}A = \frac{\sin s \sin(s-a)}{\sin b \sin c},$$

and

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}. \qquad (96)$$

In like manner,

$$\cos \tfrac{1}{2}B = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}}, \qquad (97)$$

and

$$\cos \tfrac{1}{2}C = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}. \qquad (98)$$

Dividing (93) by (96), we obtain

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \sqrt{\frac{\sin b \sin c}{\sin s \sin(s-a)}}$$

$$= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}. \qquad (99)$$

Digitized by Google

In like manner,

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}}, \qquad (100)$$

and

$$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}}. \qquad (101)$$

159. *To express the sines, cosines, and tangents of the half-sides of a spherical triangle in terms of the angles of the triangle.*

From (**90**), Art. 157, we obtain

$$\sin B \sin C \cos a = \cos A + \cos B \cos C,$$

or,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \qquad (C)$$

Then,

$$1 - \cos a = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

or,

$$2 \sin^2 \tfrac{1}{2} a = \frac{-(\cos B \cos C - \sin B \sin C) - \cos A}{\sin B \sin C}$$

$$= -\frac{\cos (B+C) + \cos A}{\sin B \sin C}.$$

Therefore by Art. 72,

$$2 \sin^2 \tfrac{1}{2} a = -\frac{2 \cos \tfrac{1}{2}(B+C+A) \cos \tfrac{1}{2}(B+C-A)}{\sin B \sin C}.$$

Denoting $A + B + C$ by $2S$, so that S is the half-sum of the angles of the triangle, we have $B + C - A = 2(S - A)$.

Whence,

$$\sin^2 \tfrac{1}{2} a = -\frac{\cos S \cos (S-A)}{\sin B \sin C},$$

and

$$\sin \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\sin B \sin C}}. \qquad (102)$$

In like manner,

$$\sin \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos (S-B)}{\sin C \sin A}}, \qquad (103)$$

and

$$\sin \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos (S-C)}{\sin A \sin B}}. \qquad (104)$$

Digitized by Google

Again, adding both members of (C) to unity, we have

$$1+\cos a = 1+\frac{\cos A+\cos B\cos C}{\sin B\sin C}$$

$$=\frac{\cos A+\cos B\cos C+\sin B\sin C}{\sin B\sin C}.$$

Then, $2\cos^2\frac{1}{2}a=\frac{\cos A+\cos(B-C)}{\sin B\sin C}$

$$=\frac{2\cos\frac{1}{2}[A+B-C]\cos\frac{1}{2}[A-(B-C)]}{\sin B\sin C},$$

or, $$\cos^2\tfrac{1}{2}a=\frac{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A-B+C)}{\sin B\sin C}.$$

But $A+B-C=2(S-C)$, and $A-B+C=2(S-B)$.

Whence,

$$\cos^2\tfrac{1}{2}a=\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C},$$

and $$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}. \qquad (105)$$

In like manner,

$$\cos\tfrac{1}{2}b=\sqrt{\frac{\cos(S-C)\cos(S-A)}{\sin C\sin A}}, \qquad (106)$$

and $$\cos\tfrac{1}{2}c=\sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}}. \qquad (107)$$

Dividing (102) by (105), we obtain

$$\tan\tfrac{1}{2}a=\sqrt{-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}. \qquad (108)$$

In like manner,

$$\tan\tfrac{1}{2}b=\sqrt{-\frac{\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)}}, \qquad (109)$$

and $$\tan\tfrac{1}{2}c=\sqrt{-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}. \qquad (110)$$

Digitized by Google

NAPIER'S ANALOGIES.

160. Dividing (**99**), Art. 158, by (**100**), we have

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}\sqrt{\frac{\sin s\sin(s-b)}{\sin(s-c)\sin(s-a)}},$$

or,
$$\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}A\sin\frac{1}{2}B}=\sqrt{\frac{\sin^2(s-b)}{\sin^2(s-a)}}=\frac{\sin(s-b)}{\sin(s-a)}.$$

Whence, by composition and division,

$$\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B+\cos\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}A\cos\frac{1}{2}B-\cos\frac{1}{2}A\sin\frac{1}{2}B}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)}.$$

That is, by Arts. 65, 66, and 73,

$$\frac{\sin(\frac{1}{2}A+\frac{1}{2}B)}{\sin(\frac{1}{2}A-\frac{1}{2}B)}=\frac{\tan\frac{1}{2}[s-b+(s-a)]}{\tan\frac{1}{2}[s-b-(s-a)]}.$$

But $s-b+s-a=2s-a-b=c.$

Hence,
$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}. \qquad (111)$$

161. Multiplying (**99**) by (**100**), we have

$$\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}},$$

or,
$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B}=\sqrt{\frac{\sin^2(s-c)}{\sin^2 s}}=\frac{\sin(s-c)}{\sin s}.$$

Whence, by composition and division,

$$\frac{\cos\frac{1}{2}A\cos\frac{1}{2}B-\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B+\sin\frac{1}{2}A\sin\frac{1}{2}B}=\frac{\sin s-\sin(s-c)}{\sin s+\sin(s-c)},$$

or, by Art. 73,
$$\frac{\cos(\frac{1}{2}A+\frac{1}{2}B)}{\cos(\frac{1}{2}A-\frac{1}{2}B)}=\frac{\tan\frac{1}{2}[s-(s-c)]}{\tan\frac{1}{2}[s+(s-c)]}.$$

But $s+s-c=2s-c=a+b.$

Hence,
$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}. \qquad (112)$$

Digitized by Google

162. Proceeding as in Art. 157, and applying (111) to the triangle $A'B'C'$, we obtain,

$$\frac{\sin\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'-b')}.$$

But, $\frac{1}{2}(A'+B')=\frac{1}{2}(180°-a+180°-b)$

$=180°-\frac{1}{2}(a+b)$,

$\frac{1}{2}(A'-B')=\frac{1}{2}(180°-a-180°+b)=\frac{1}{2}(-a+b)$,

$\frac{1}{2}c'=\frac{1}{2}(180°-C)=90°-\frac{1}{2}C$,

and $\frac{1}{2}(a'-b')=\frac{1}{2}(180°-A-180°+B)=\frac{1}{2}(-A+B)$.

Whence,

$$\frac{\sin[180°-\frac{1}{2}(a+b)]}{\sin\frac{1}{2}(-a+b)}=\frac{\tan(90°-\frac{1}{2}C)}{\tan\frac{1}{2}(-A+B)}.$$

Therefore, by Arts. 42, 46, and 47,

$$\frac{\sin\frac{1}{2}(a+b)}{-\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{-\tan\frac{1}{2}(A-B)},$$

or,
$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}. \qquad (113)$$

In like manner, from (112), we obtain

$$\frac{\cos\frac{1}{2}(A'+B')}{\cos\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'+b')}.$$

But, $\frac{1}{2}(a'+b')=180°-\frac{1}{2}(A+B)$.

Whence,

$$\frac{\cos[180°-\frac{1}{2}(a+b)]}{\cos\frac{1}{2}(-a+b)}=\frac{\tan(90°-\frac{1}{2}C)}{\tan[180°-\frac{1}{2}(A+B)]}.$$

Therefore, by Arts. 42, 46, and 47,

$$\frac{-\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{-\tan\frac{1}{2}(A+B)},$$

or,
$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}. \qquad (114)$$

Digitized by Google

163. The formulæ exemplified in Arts. 160, 161, and 162, are known as *Napier's Analogies*. In each case there may be other forms, according as other elements are used.

SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

164. In the solution of spherical oblique triangles, we may distinguish six cases:

1. *Given a side and two adjacent angles.*
2. *Given two sides and their included angle.*
3. *Given the three sides.*
4. *Given the three angles.*
5. *Given two sides and the angle opposite to one of them.*
6. *Given two angles and the side opposite to one of them.*

165. By application of the principles of Art. 136, (f), the solution of any example under Cases 2, 4, and 6 may be made to depend upon the solution of another example under Cases 1, 3, and 5, respectively; and *vice versa*.

Thus it is not essential to consider more than three cases in the solution of spherical oblique triangles.

166. The student must carefully bear in mind the remarks made in Arts. 151 and 152.

Case I.

167. *Given a side and two adjacent angles.*

1. Given $A = 70°$, $B = 131° 20'$, $c = 116°$; find a, b, and C.

By Napier's Analogies (Arts. 160, 161), we have

$$\frac{\sin \frac{1}{2}(B+A)}{\sin \frac{1}{2}(B-A)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(b-a)},$$

and

$$\frac{\cos \frac{1}{2}(B+A)}{\cos \frac{1}{2}(B-A)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(b+a)}.$$

Digitized by Google

Whence,

$$\tan \tfrac{1}{2}(b-a) = \sin \tfrac{1}{2}(B-A) \csc \tfrac{1}{2}(B+A) \tan \tfrac{1}{2}c,$$

$$\text{and} \quad \tan \tfrac{1}{2}(b+a) = \cos \tfrac{1}{2}(B-A) \sec \tfrac{1}{2}(B+A) \tan \tfrac{1}{2}c.$$

From the data,

$$\tfrac{1}{2}(B-A) = 30^\circ\,40', \quad \tfrac{1}{2}(B+A) = 100^\circ\,40', \quad \tfrac{1}{2}c = 58^\circ.$$

$$\log \sin \tfrac{1}{2}(B-A) = 9.7076 \qquad \log \cos \tfrac{1}{2}(B-A) = 9.9346$$
$$\log \csc \tfrac{1}{2}(B+A) = 0.0076 \qquad \log \sec \tfrac{1}{2}(B+A) = 0.7326$$
$$\log \tan \tfrac{1}{2}c = 0.2042 \qquad \log \tan \tfrac{1}{2}c = 0.2042$$
$$\log \tan \tfrac{1}{2}(b-a) = 9.9194 \qquad \log \tan \tfrac{1}{2}(b+a) = 0.8714$$
$$\therefore \tfrac{1}{2}(b-a) = 39^\circ\,42.8'. \qquad 180^\circ - \tfrac{1}{2}(b+a) = 82^\circ\,20.5'$$
$$\therefore \tfrac{1}{2}(b+a) = 97^\circ\,39.5'.$$

Then, $a = \tfrac{1}{2}(b+a) - \tfrac{1}{2}(b-a) = 57^\circ\,56.7'$,

and $b = \tfrac{1}{2}(b+a) + \tfrac{1}{2}(b-a) = 137^\circ\,22.3'$.

To find C, we have by Art. 162,

$$\cot \tfrac{1}{2}C = \frac{\sin \tfrac{1}{2}(b+a)}{\sin \tfrac{1}{2}(b-a)} \tan \tfrac{1}{2}(B-A)$$
$$= \sin \tfrac{1}{2}(b+a) \csc \tfrac{1}{2}(b-a) \tan \tfrac{1}{2}(B-A).$$

$$\log \sin \tfrac{1}{2}(b+a) = 9.9961$$
$$\log \csc \tfrac{1}{2}(b-a) = 0.1946$$
$$\log \tan \tfrac{1}{2}(B-A) = 9.7730$$
$$\log \cot \tfrac{1}{2}C = 9.9637$$
$$\therefore \tfrac{1}{2}C = 47^\circ\,23.6', \text{ and } C = 94^\circ\,47.2'.$$

Note 1. The value of C may also be determined by the formula

$$\cot \tfrac{1}{2}C = \frac{\cos \tfrac{1}{2}(b+a)}{\cos \tfrac{1}{2}(b-a)} \tan \tfrac{1}{2}(B+A) \text{ (Art. 162).}$$

Note 2. The triangle is always possible for any values of the given elements.

Digitized by Google

EXAMPLES.

Solve the following triangles:

2. Given $A = 78°$, $B = 41°$, $c = 108°$.

3. Given $B = 135°$, $C = 50°$, $a = 70° 20'$.

4. Given $A = 31° 40'$, $C = 122° 20'$, $b = 40° 40'$.

5. Given $A = 108° 12'$, $B = 145° 46'$, $c = 126° 32'$.

CASE II.

168. *Given two sides and their included angle.*

1. Given $b = 137° 20'$, $c = 116°$, $A = 70°$; find B, C, and a.

By Napier's Analogies (Art. 162),

$$\frac{\sin \frac{1}{2}(b+c)}{\sin \frac{1}{2}(b-c)} = \frac{\cot \frac{1}{2}A}{\tan \frac{1}{2}(B-C)},$$

and

$$\frac{\cos \frac{1}{2}(b+c)}{\cos \frac{1}{2}(b-c)} = \frac{\cot \frac{1}{2}A}{\tan \frac{1}{2}(B+C)}.$$

Whence,

$$\tan \tfrac{1}{2}(B-C) = \sin \tfrac{1}{2}(b-c) \csc \tfrac{1}{2}(b+c) \cot \tfrac{1}{2}A,$$

(−, +, −, +)

and

$$\tan \tfrac{1}{2}(B+C) = \cos \tfrac{1}{2}(b-c) \sec \tfrac{1}{2}(b+c) \cot \tfrac{1}{2}A.$$

From the data,

$$\tfrac{1}{2}(b-c) = 10° 40', \quad \tfrac{1}{2}(b+c) = 126° 40', \quad \tfrac{1}{2}A = 35°.$$

From which we find,

$$\tfrac{1}{2}(B-C) = 18° 14.5', \quad \tfrac{1}{2}(B+C) = 113° 2.9'.$$

Then, $B = \frac{1}{2}(B+C) + \frac{1}{2}(B-C) = 131° 17.4'$,

and $C = \frac{1}{2}(B+C) - \frac{1}{2}(B-C) = 94° 48.4'$.

To find a, we have by Art. 160,

$$\tan \tfrac{1}{2}a = \frac{\sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B-C)} \tan \tfrac{1}{2}(b-c).$$

From which we obtain $a = 57° 56.6'$.

Digitized by Google

Note. The triangle is always possible for any values of the given elements.

EXAMPLES.

Solve the following triangles:

2. Given $a = 72°$, $b = 47°$, $C = 33°$.

3. Given $a = 98°$, $c = 60°$, $B = 110°$.

4. Given $b = 120°\ 20'$, $c = 70°\ 40'$, $A = 50°$.

5. Given $a = 125°\ 10'$, $b = 153°\ 50'$, $C = 140°\ 20'$.

Case III.

169. *Given the three sides.*

1. Given $a = 60°$, $b = 137°\ 20'$, $c = 116°$; find A, B, and C.

By Art. 158,

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}},$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}},$$

and $$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}}.$$

From the data,

$$2s = a + b + c = 313°\ 20', \text{ or } s = 156°\ 40'.$$

Whence,

$$s - a = 96°\ 40',\ s - b = 19°\ 20',\ s - c = 40°\ 40'.$$

$$\log \sin (s-b) = 9.5199$$
$$\log \sin (s-c) = 9.8140$$
$$\log \csc s = 0.4022$$
$$\log \csc (s-a) = 0.0029$$
$$2 \overline{)9.7390}$$
$$\log \tan \tfrac{1}{2} A = 9.8695$$
$$\therefore\ \tfrac{1}{2} A = 36°\ 31.2', \text{ and } A = 73°\ 2.4'.$$

Digitized by Google

In like manner we find,

$$B = 131° 32.2', \text{ and } C = 96° 55.4'.$$

The values of A, B, and C may also be obtained by aid of the sine or cosine formulæ of Art. 158.

If all the angles are to be computed, the tangent formulæ are the most convenient, as only four different logarithms are required. If but one angle is required, the cosine formulæ will be found to involve the least work.

Note. The triangle is always possible for any values of the given elements which satisfy the conditions of Art. 138, (a) and (c); that is, if $a+b+c$ is $<360°$, and no side is greater than the sum of the other two.

EXAMPLES.

Solve the following triangles:

2. Given $a = 38°$, $b = 51°$, $c = 42°$.

3. Given $a = 101°$, $b = 49°$, $c = 60°$.

4. Given $a = 61°$, $b = 39°$, $c = 92°$.

5. Given $a = 62° 20'$, $b = 54° 10'$, $c = 97° 50'$.

CASE IV.

170. *Given the three angles.*

1. Given $A = 70°$, $B = 131° 10'$, $C = 94° 50'$; find a, b, and c.

By Art. 159,

$$\tan \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}},$$

$$\tan \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos (S-B)}{\cos (S-C) \cos (S-A)}},$$

and

$$\tan \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos (S-C)}{\cos (S-A) \cos (S-B)}}.$$

Digitized by Google

From the data,

$$2S = A + B + C = 296°, \text{ or } S = 148°;$$
$$S - A = 78°,\ S - B = 16°\,50',\ S - C = 53°\,10'.$$

Note 1. Since $\cos S$ is $-$ (Art. 37), while the cosines of $S - A$, $S - B$, and $S - C$ are $+$, the quantities under the radical signs are essentially positive, and hence no attention need be paid to the negative signs in the formulæ.

$$\begin{aligned}
\log \cos S &= 9.9284 \\
\log \cos (S - A) &= 9.3179 \\
\log \sec (S - B) &= 0.0190 \\
\log \sec (S - C) &= 0.2222 \\
2\overline{)9.4875} & \\
\log \tan \tfrac{1}{2}a &= 9.7437 \\
\therefore\ \tfrac{1}{2}a &= 28°\,59.7', \text{ and } a = 57°\,59.4'.
\end{aligned}$$

In like manner we find,

$$b = 137°\,11.8',\ \text{and}\ c = 115°\,55.8'.$$

The values of a, b, and c may also be obtained by aid of the sine or cosine formulæ of Art. 159.

If all the sides are to be computed, the tangent formulæ are the most convenient, as only four different logarithms are required. If but one angle is required, the sine formulæ will be found to involve the least work.

Note 2. The triangle is always possible for any values of the given elements, provided S is between 90° and 270°, and each of the quantities $S - A$, $S - B$, and $S - C$ between 90° and $-90°$ (Art. 37).

EXAMPLES.

Solve the following triangles:

2. Given $A = 75°$, $B = 82°$, $C = 61°$.

3. Given $A = 120°$, $B = 130°$, $C = 80°$.

4. Given $A = 91°\,10'$, $B = 85°\,40'$, $C = 72°\,30'$.

5. Given $A = 138°\,16'$, $B = 31°\,11'$, $C = 35°\,53'$.

Digitized by Google

Case V.

171. *Given two sides and the angle opposite to one of them.*

1. Given $a = 58°$, $b = 137° 20'$, $B = 131° 20'$; find A, C, and c.

By Art. 155, $$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b},$$

or, $$\sin A = \sin a \csc b \sin B.$$

$$\log \sin a = 9.9284$$
$$\log \csc b = 0.1689$$
$$\log \sin B = 9.8756$$
$$\log \sin A = 9.9729$$
$$\therefore A = 69° 58', \text{ or } 110° 2' \text{ (Art. 152)}.$$

To find C and c, we have by Arts. 160 and 162,

$$\cot \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(b+a)}{\sin \tfrac{1}{2}(b-a)} \tan \tfrac{1}{2}(B-A),$$

and $$\tan \tfrac{1}{2} c = \frac{\sin \tfrac{1}{2}(B+A)}{\sin \tfrac{1}{2}(B-A)} \tan \tfrac{1}{2}(b-a).$$

Using the first value of A, we have

$$\tfrac{1}{2}(B+A) = 100° 39', \quad \tfrac{1}{2}(B-A) = 30° 41'.$$

Also, $$\tfrac{1}{2}(b+a) = 97° 40', \quad \tfrac{1}{2}(b-a) = 39° 40'.$$

From which we obtain

$$C = 94° 41.6', \text{ and } c = 115° 53.6'.$$

Using the second value of A, we have

$$\tfrac{1}{2}(B+A) = 120° 41', \quad \tfrac{1}{2}(B-A) = 10° 39'.$$

From which we obtain

$$C = 147° 26.4', \text{ and } c = 150° 56.8'.$$

Thus the two solutions are:

1. $A = 69° 58'$, $C = 94° 41.6'$, $c = 115° 53.6'$.
2. $A = 110° 2'$, $C = 147° 26.4'$, $c = 150° 56.8'$.

Digitized by Google

As in the corresponding case in the solution of plane oblique triangles (compare Arts. 124 to 126), there may sometimes be two solutions, sometimes only one, and sometimes none, in an example under Case V.

After the two values of A have been obtained, the number of solutions may be readily determined by inspection; for by Art. 136, (b), if a is $< b$, A must be $< B$, and if a is $> b$, A must be $> B$.

That is, *only those values of A can be retained which are greater or less than B according as a is greater or less than b.*

Thus, in Ex. 1, a is given $< b$; and since both values of A, $69° 58'$ and $110° 2'$, are $< B$, we have two solutions.

Also if the data are such as to make $\log \sin A$ positive, there will be no solution corresponding.

2. Given $a = 58°$, $c = 116°$, $C = 94° 50'$; find A, B, and b.

In this case, $$\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c},$$

or, $$\sin A = \sin a \csc c \sin C.$$

$$\log \sin a = 9.9284$$
$$\log \csc c = 0.0463$$
$$\log \sin C = 9.9985$$
$$\log \sin A = 9.9732$$
$$\therefore A = 70° 5', \text{ or } 109° 55'.$$

Since a is given $< c$, only values of A which are $< C$ can be retained; hence there is but one solution, corresponding to the *acute* value of A.

To find B and b, we have by Arts. 160 and 162,

$$\cot \tfrac{1}{2} B = \frac{\sin \tfrac{1}{2}(c + a)}{\sin \tfrac{1}{2}(c - a)} \tan \tfrac{1}{2}(C - A),$$

and $$\tan \tfrac{1}{2} b = \frac{\sin \tfrac{1}{2}(C + A)}{\sin \tfrac{1}{2}(C - A)} \tan \tfrac{1}{2}(c - a).$$

Digitized by Google

Using the first value of A, we have

$$\tfrac{1}{2}(C+A) = 82^\circ\ 27.5', \quad \tfrac{1}{2}(C-A) = 12^\circ\ 22.5';$$

also, $\tfrac{1}{2}(c+a) = 87^\circ$, $\tfrac{1}{2}(c-a) = 29^\circ$.

From which we obtain

$$B = 131^\circ\ 21.8', \text{ and } b = 137^\circ\ 23.6'.$$

Thus the only solution is

$$A = 70^\circ\ 5', \ B = 131^\circ\ 21.8', \ b = 137^\circ\ 23.6'.$$

3. Given $b = 126^\circ$, $c = 70^\circ$, $B = 56^\circ$; find C.

In this case, $\dfrac{\sin C}{\sin B} = \dfrac{\sin c}{\sin b}$,

or, $\sin C = \sin c \csc b \sin B$.

$$\begin{aligned} \log \sin c &= 9.9730 \\ \log \csc b &= 0.0920 \\ \log \sin B &= 9.9186 \\ \hline \log \sin C &= 9.9836 \end{aligned}$$

$\therefore C = 74^\circ\ 20'$, or $105^\circ\ 40'$.

Since both values of C are $> B$, while c is given $< b$, there is no solution.

EXAMPLES.

Solve the following triangles:

4. Given $b = 99^\circ\ 40'$, $c = 64^\circ\ 20'$, $B = 95^\circ\ 40'$.

5. Given $a = 40^\circ$, $b = 118^\circ\ 20'$, $A = 29^\circ\ 40'$.

6. Given $a = 115^\circ\ 20'$, $c = 146^\circ\ 20'$, $C = 141^\circ\ 10'$.

7. Given $a = 109^\circ\ 20'$, $c = 82^\circ$, $A = 107^\circ\ 40'$.

8. Given $b = 108^\circ\ 30'$, $c = 40^\circ\ 50'$, $C = 39^\circ\ 50'$.

9. Given $a = 162^\circ\ 20'$, $b = 15^\circ\ 40'$, $B = 125^\circ$.

10. Given $a = 55^\circ$, $c = 138^\circ\ 10'$, $A = 42^\circ\ 30'$.

Digitized by Google

Case VI.

172. *Given two angles and the side opposite to one of them.*

1. Given $A = 110°$, $B = 131° 20'$, $b = 137° 20'$; find a, c, and C.

In this case,
$$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B},$$

or,
$$\sin a = \sin A \csc B \sin b.$$

$$\log \sin A = 9.9730$$
$$\log \csc B = 0.1244$$
$$\log \sin b = 9.8311$$
$$\log \sin a = 9.9285$$
$$\therefore a = 58° 1.2', \text{ or } 121° 58.8'.$$

To find c and C, we have by Arts. 160 and 162,

$$\tan \tfrac{1}{2}c = \frac{\sin \tfrac{1}{2}(B+A)}{\sin \tfrac{1}{2}(B-A)} \tan \tfrac{1}{2}(b-a),$$

and
$$\cot \tfrac{1}{2}C = \frac{\sin \tfrac{1}{2}(b+a)}{\sin \tfrac{1}{2}(b-a)} \tan \tfrac{1}{2}(B-A).$$

Using the first value of a, we have

$$c = 150° 53.8', \text{ and } C = 147° 23';$$

and using the second value of a,

$$c = 64° 7.8', \text{ and } C = 85° 17.6'.$$

Thus the two solutions are:

1. $a = 58° 1.2'$, $c = 150° 53.8'$, $C = 147° 23'$.
2. $a = 121° 58.8'$, $c = 64° 7.8'$, $C = 85° 17.6'$.

In Case VI., as well as in Case V., there are sometimes two solutions, sometimes only one, and sometimes none; and it may be shown, exactly as in Art. 171, that *only those values of a can be retained which are greater or less than b according as A is greater or less than B.*

Digitized by Google

Also if log sin a is positive, the triangle is impossible.

EXAMPLES.

Solve the following triangles:

2. Given $B = 116°$, $C = 80°$, $c = 84°$.

3. Given $A = 132°$, $B = 140°$, $b = 127°$.

4. Given $A = 62°$, $C = 102°$, $a = 64° 30'$.

5. Given $A = 133° 50'$, $B = 66° 30'$, $a = 81° 10'$.

6. Given $B = 22° 20'$, $C = 146° 40'$, $c = 138° 20'$.

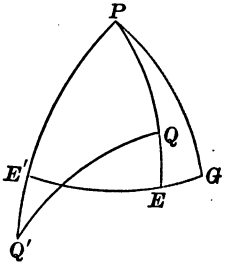
7. Given $A = 61° 40'$, $C = 140° 20'$, $c = 150° 20'$.

8. Given $B = 73°$, $C = 81° 20'$, $b = 122° 40'$.

APPLICATIONS.

173. In questions concerning geodesy or navigation, the earth may be regarded as a sphere.

The *shortest path* between any two points is the arc of a great circle which joins them, and the angles between this arc and the meridians of the points determine the bearings of the points from each other.



Thus, if Q and Q' are the points, and PQ and PQ' their meridians, the angle PQQ' determines the bearing of Q' from Q, and the angle $PQ'Q$ determines the bearing of Q from Q'.

Digitized by Google

If the latitudes and longitudes of Q and Q' are known, the arc QQ' and the angles PQQ' and $PQ'Q$ may be determined by the solution of a spherical triangle.

For if EE' is the equator, and PG the meridian of Greenwich, we have

$$\text{angle } QPQ' = \text{angle } Q'PG - \text{angle } QPG$$
$$= \text{longitude of } Q' - \text{longitude of } Q.$$

Also, $PQ = PE - QE = 90° - \text{latitude of } Q,$

and $PQ' = PE' + Q'E' = 90° + \text{latitude of } Q'.$

Thus, in the triangle PQQ', two sides and their included angle are known, and the remaining elements may be computed.

Note. When QQ' has been found in angular measure, its length in miles may be calculated by the method of Art. 132. In the following problems the diameter of the earth is taken as 7912 miles.

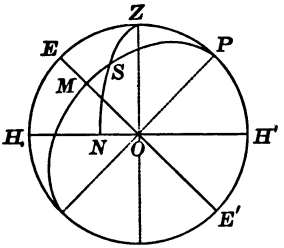
1. Boston lies in lat. 42° 21′ N., longitude 71° 4′ W.; and the latitude of Greenwich is 51° 29′ N. Find the shortest distance in miles between the places, and the bearing of each place from the other.

2. Calcutta lies in lat. 22° 33′ N., lon. 88° 19′ E.; and Valparaiso lies in lat. 33° 2′ S., lon. 71° 42′ W. Find the shortest distance in miles between the places, and the bearing of each place from the other.

3. Sandy Hook lies in lat. 40° 28′ N., lon. 74° 1′ W.; and Queenstown lies in lat. 51° 50′ N., lon. 8° 19′ W. In what latitude does a great circle course from Sandy Hook to Queenstown cross the meridian of 50° W.?

If the latitude of a place is known, and the altitude and declination of the sun, the solution of a spherical triangle serves to determine the hour of the day at the time and place of observation.

Digitized by Google



Thus let O be the position of the observer; P the celestial north pole; EE' the celestial equator; HH' the horizon; Z the zenith; S the sun's position; PSM a meridian passing through the sun's position; and ZSN a great circle passing through Z and S.

Then SM is the sun's declination, SN its altitude, and EZ the latitude of the place of observation.

Then in the spherical triangle SPZ, we have

$$SP = PM - SM = 90° - \text{the sun's declination},$$

$$SZ = ZN - SN = 90° - \text{the sun's altitude},$$

and $$PZ = EP - EZ = 90° - \text{the latitude of the place}.$$

That is, the three sides of the triangle SPZ are known, and the angle SPZ may be computed.

If 24 hours is multiplied by the ratio of this angle to 360°, we have the time required for the sun to move from S to the meridian EP.

Hence, if this time is subtracted from 12 o'clock, if the observation is made in the morning, or added, if made in the afternoon, we obtain the hour of the day at the time and place of observation.

If the Greenwich time of the observation is noted on a chronometer, the difference between this and the local time as calculated above serves to determine the longitude of the place of observation.

Digitized by Google

In reducing time to longitude, it should be borne in mind that 24 hours of time correspond to 360° of longitude; that is, one hour of time corresponds to 15° of longitude, one minute to 15′, and one second to 15″.

4. A mariner observes the altitude of the sun to be 14° 18′, its declination being 18° 36′ N. If the latitude of the vessel is 50° 13′ N., and the observation is made in the morning, find the hour of the day. If the observation is taken at 9 A.M., Greenwich time, what is the longitude of the vessel?

5. What will be the altitude of the sun at 4 P.M. in San Francisco, lat. 37° 48′ N., its declination being 12° S.?

6. In Melbourne, lat. 37° 49′ S., the altitude of the sun is observed to be 25° 46′. If the sun's declination is 3° S., and the observation is made in the morning, find the hour of the day.

7. At what hour will the sun rise in Boston, lat. 42° 21′ N., when its declination is 15° N.?

Note. At sunrise the sun's altitude is 0, so that the arc SZ becomes 90°.

Digitized by Google

FORMULÆ.

PLANE TRIGONOMETRY.

Art. 10.
$$\left.\begin{aligned} &\sin A = \frac{a}{c}, \quad \tan A = \frac{a}{b}, \quad \sec A = \frac{c}{b}, \\ &\cos A = \frac{b}{c}, \quad \cot A = \frac{b}{a}, \quad \csc A = \frac{c}{a}. \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} &\sin B = \frac{b}{c}, \quad \tan B = \frac{b}{a}, \quad \sec B = \frac{c}{a}, \\ &\cos B = \frac{a}{c}, \quad \cot B = \frac{a}{b}, \quad \csc B = \frac{c}{b}. \end{aligned}\right\} \quad (2)$$

Art. 18.
$$\left.\begin{aligned} &\sin A = \frac{1}{\csc A}, \quad \tan A = \frac{1}{\cot A}, \quad \sec A = \frac{1}{\cos A}, \\ &\cos A = \frac{1}{\sec A}, \quad \cot A = \frac{1}{\tan A}, \quad \csc A = \frac{1}{\sin A}. \end{aligned}\right\} \quad (3)$$

Art. 19.
$$\sin^2 A + \cos^2 A = 1. \quad (4)$$

Art. 20.
$$\tan A = \frac{\sin A}{\cos A}. \quad (5) \qquad \cot A = \frac{\cos A}{\sin A}. \quad (6)$$

Art. 21.
$$\sec^2 A = 1 + \tan^2 A. \quad (7) \qquad \csc^2 A = 1 + \cot^2 A. \quad (8)$$

Art. 42.
$$\left.\begin{aligned} &\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \\ &\tan(-A) = -\tan A, \quad \cot(-A) = -\cot A, \\ &\sec(-A) = \sec A, \quad \csc(-A) = -\csc A. \end{aligned}\right\} \quad (9)$$

Art. 44.
$$\left.\begin{aligned} &\sin(90°+A) = \cos A, \quad \cos(90°+A) = -\sin A, \\ &\tan(90°+A) = -\cot A, \quad \cot(90°+A) = -\tan A, \\ &\sec(90°+A) = -\csc A, \quad \csc(90°+A) = \sec A. \end{aligned}\right\} \quad (10)$$

Digitized by Google

Art. 65. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$ (11)

$\cos(x+y) = \cos x \cos y - \sin x \sin y.$ (12)

Art. 66. $\sin(x-y) = \sin x \cos y - \cos x \sin y.$ (13)

$\cos(x-y) = \cos x \cos y + \sin x \sin y.$ (14)

Art. 71. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$ (15)

$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$ (16)

$\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}.$ (17)

$\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$ (18)

Art. 72. $\sin x + \sin y = 2 \sin \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y).$ (19)

$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y).$ (20)

$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y).$ (21)

$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y).$ (22)

Art. 73. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{1}{2}(x+y)}{\tan \frac{1}{2}(x-y)}.$ (23)

Art. 74. $\sin 2x = 2 \sin x \cos x.$ (24)

$\cos 2x = \cos^2 x - \sin^2 x.$ (25)

$\cos 2x = 1 - 2 \sin^2 x.$ (26)

$\cos 2x = 2 \cos^2 x - 1.$ (27)

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}.$ (28)

$\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}.$ (29)

Digitized by Google

Art. 75.

$$\sin\tfrac{1}{2}x=\sqrt{\frac{1-\cos x}{2}}. \quad (30) \qquad \cos\tfrac{1}{2}x=\sqrt{\frac{1+\cos x}{2}}. \quad (31)$$

$$\tan\tfrac{1}{2}x=\sqrt{\frac{1-\cos x}{1+\cos x}}. \quad (32)$$

$$\tan\tfrac{1}{2}x=\frac{\sin x}{1+\cos x}. \quad (33) \qquad \tan\tfrac{1}{2}x=\frac{1-\cos x}{\sin x}. \quad (34)$$

$$\cot\tfrac{1}{2}x=\frac{1+\cos x}{\sin x}=\frac{\sin x}{1-\cos x}. \quad (35)$$

Art. 111.

$$4K=c^2\sin 2A. \quad (36) \qquad 4K=c^2\sin 2B. \quad (37)$$

$$2K=a^2\cot A. \quad (38) \qquad 2K=b^2\cot B. \quad (39)$$

$$2K=a^2\tan B. \quad (40) \qquad 2K=b^2\tan A. \quad (41)$$

$$2K=a\sqrt{(c+a)(c-a)}. \quad (42) \qquad 2K=b\sqrt{(c+b)(c-b)}. \quad (43)$$

$$2K=ab. \quad (44)$$

Art. 113.

$$\frac{\sin A}{\sin B}=\frac{a}{b}. \quad (45) \qquad \frac{\sin B}{\sin C}=\frac{b}{c}. \quad (46) \qquad \frac{\sin C}{\sin A}=\frac{c}{a}. \quad (47)$$

Art. 114.

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}. \quad (48)$$

$$\frac{b+c}{b-c}=\frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)}. \quad (49)$$

$$\frac{c+a}{c-a}=\frac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)}. \quad (50)$$

Art. 115.

$$\frac{a+b}{a-b}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}. \quad (51)$$

Art. 116.

$$a^2=b^2+c^2-2bc\cos A. \quad (52)$$

$$b^2=c^2+a^2-2ca\cos B. \quad (53)$$

$$c^2=a^2+b^2-2ab\cos C. \quad (54)$$

Digitized by Google

Art. 117.
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad (55)$$
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}. \quad (56)$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \quad (57)$$

Art. 118.
$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}. \quad (58)$$
$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{ca}}. \quad (59)$$
$$\sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}. \quad (60)$$
$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}. \quad (61)$$
$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}. \quad (62)$$
$$\cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}. \quad (63)$$
$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad (64)$$
$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}. \quad (65)$$
$$\tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \quad (66)$$

Art. 119.
$$2K = bc \sin A. \quad (67)$$
$$2K = ca \sin B. \quad (68)$$
$$2K = ab \sin C. \quad (69)$$
$$2K = \frac{a^2 \sin B \sin C}{\sin A}. \quad (70)$$
$$2K = \frac{b^2 \sin C \sin A}{\sin B}. \quad (71)$$
$$2K = \frac{c^2 \sin A \sin B}{\sin C}. \quad (72)$$
$$K = \sqrt{s(s-a)(s-b)(s-c)}. \quad (73)$$

Digitized by Google

SPHERICAL TRIGONOMETRY.

Art. 139.

$$\cos c = \cos a \cos b. \tag{74}$$

$$\sin A = \frac{\sin a}{\sin c}. \tag{75}$$

$$\cos A = \frac{\tan b}{\tan c}. \tag{76}$$

$$\sin B = \frac{\sin b}{\sin c}. \tag{77}$$

$$\cos B = \frac{\tan a}{\tan c}. \tag{78}$$

Art. 140.

$$\tan A = \frac{\tan a}{\sin b}. \tag{79}$$

$$\tan B = \frac{\tan b}{\sin a}. \tag{80}$$

Art. 141.

$$\sin A = \frac{\cos B}{\cos b}. \tag{81}$$

$$\sin B = \frac{\cos A}{\cos a}. \tag{82}$$

Art. 142.

$$\cos c = \cot A \cot B. \tag{83}$$

Art. 155.

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}. \tag{84}$$

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}. \tag{85}$$

$$\frac{\sin C}{\sin A} = \frac{\sin c}{\sin a}. \tag{86}$$

Art. 156.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{87}$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B. \tag{88}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{89}$$

Art. 157.

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \tag{90}$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b. \tag{91}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{92}$$

Digitized by Google

Art. 158. $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}. \quad (93)$$

$$\sin \tfrac{1}{2} B = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}}. \quad (94)$$

$$\sin \tfrac{1}{2} C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}. \quad (95)$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}. \quad (96)$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}}. \quad (97)$$

$$\cos \tfrac{1}{2} C = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}. \quad (98)$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}. \quad (99)$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}}. \quad (100)$$

$$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}. \quad (101)$$

Art. 159. $$\sin \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos(S-A)}{\sin B \sin C}}. \quad (102)$$

$$\sin \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos(S-B)}{\sin C \sin A}}. \quad (103)$$

$$\sin \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos(S-C)}{\sin A \sin B}}. \quad (104)$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}. \quad (105)$$

$$\cos \tfrac{1}{2} b = \sqrt{\frac{\cos(S-C)\cos(S-A)}{\sin C \sin A}}. \quad (106)$$

Digitized by Google

$$\cos \tfrac{1}{2} c = \sqrt{\frac{\cos (S-A) \cos (S-B)}{\sin A \sin B}}. \quad (107)$$

$$\tan \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}. \quad (108)$$

$$\tan \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos (S-B)}{\cos (S-C) \cos (S-A)}}. \quad (109)$$

$$\tan \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos (S-C)}{\cos (S-A) \cos (S-B)}}. \quad (110)$$

Art. 160.

$$\frac{\sin \frac{1}{2} (A+B)}{\sin \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a-b)}. \quad (111)$$

Art. 161.

$$\frac{\cos \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a+b)}. \quad (112)$$

Art. 162.

$$\frac{\sin \frac{1}{2} (a+b)}{\sin \frac{1}{2} (a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A-B)}. \quad (113)$$

$$\frac{\cos \frac{1}{2} (a+b)}{\cos \frac{1}{2} (a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A+B)}. \quad (114)$$

Digitized by Google

ANSWERS.

Art. 9; page 3.

9. 28° 38′ 52.4″. **14.** 114° 35′ 29.6″.
13. 42° 58′ 18.6″. **15.** 100° 54′ 5.1″.
16. 30° 40′ 33.8″.

Art. 91; page 57.

2. .7781.	**7.** 1.3222.	**12.** 1.9912.	**17.** 2.1303.
3. 1.1461.	**8.** 1.7993.	**13.** 2.0212.	**18.** 2.2252.
4. .9030.	**9.** 1.7481.	**14.** 2.0491.	**19.** 2.1673.
5. 1.0791.	**10.** 1.9242.	**15.** 2.1582.	**20.** 2.5741.
6. 1.1761.	**11.** 1.6532.	**16.** 2.3343.	**21.** 2.5353.

Art. 93; page 58.

2. .3680.	**5.** 1.5441.	**8.** .2252.	**11.** .8539.
3. .1549.	**6.** .1182.	**9.** 2.2431.	**12.** .7660.
4. .5229.	**7.** 2.0970.	**10.** 1.0458.	**13.** .7360.

Art. 96; page 59.

3. .2863.	**9.** 4.5844.	**15.** .1165.	**22.** .2601.
4. 2.7090.	**10.** 3.2620.	**16.** .3860.	**23.** .6884.
5. 4.2255.	**11.** .9801.	**17.** .2212.	**24.** .1840.
6. .1398.	**12.** .4225.	**18.** .1750.	**25.** .2215.
7. .7194.	**13.** .1590.	**20.** 2.6145.	**26.** .2494.
8. .6611.	**14.** .0430.	**21.** .1678.	**27.** .1449.

Digitized by Google

Art. 98; page 60.

2. .2552.
3. .3522.
4. 9.2922 − 10.
5. 8.6811 − 10.
6. 1.5841.
7. 7.7323 − 10.
8. 6.4983 − 10.
9. 3.8663.
10. .6074.
11. 9.6511 − 10.
12. 2.4804.
13. 8.7905 − 10.
14. 6.3588.
15. .1964.
16. .1688.

Art. 105; pages 65 and 66.

1. 8.454.
2. 10.73.
3. − 2202.
4. .2179.
5. .01157.
6. − .7032.
7. 7.672.
8. .6688.
9. − 3.908.
10. 1782.
11. .3500.
12. − .4748.
13. .4127.
14. − 4.671.
15. .2415.
16. − .0725.
17. 13587.
18. .006415.
19. − 1.184.
20. .000007038.
21. 2.924.
22. .9146.
23. 4.638.
24. .0000639.
25. 1.414.
26. 1.495.
27. − 1.246.
28. .6553.
29. .2846.
30. 2.372.
31. − .5142.
32. .1588.
35. 5.883.
36. .7885.
37. 1.195.
38. .6803.
39. .6443.
40. .5010.
41. 1.062.
42. − .9102.
43. 1.093.
44. .7035.
45. .5807.
46. − .6313.
47. 24.62.
48. .2979.
49. 98.50.
50. 1.660.
51. 3.076.
52. .8678.
53. 1.134.
54. .5881.
55. 1.805.
56. .003229.
57. .03344.

Art. 110; pages 69 to 72.

1. $a = 7.708$, $b = 8.124$.
2. $b = 1883$, $c = 2019.5$.
3. $a = 24.37'$, $c = 53.56$.
4. $A = 43° 17.9'$, $b = .6622$.

Digitized by Google

5. $A = 68° 12.2'$, $c = 5.385$.

6. $b = 26.91$, $c = 87.64$.

7. $a = .02309$, $b = .01452$.

8. $A = 55° 36.1'$, $a = 4.216$.

9. $a = 5571$, $c = 7007$.

10. $A = 41° 2.4'$, $c = 153.8$.

11. $b = 167.6$, $c = 230.5$.

12. $a = 30.51$, $b = 18.59$.

13. $A = 47° 52.5'$, $b = 184.7$.

14. $a = 2.847$, $c = 3.287$.

15. $a = 5.4125$, $c = 14.306$.

16. $a = .7133$, $b = .1367$.

17. $A = 51° 51.9'$, $c = 811.7$.

18. $b = 42.27$, $c = 208.15$.

19. $A = 58° 35.7'$, $a = .0409$.

20. $b = 76.13$, $c = 1877$.

21. $A = 36° 45.9'$, $c = 41.22$.

22. $A = 65° 30'$, $a = 3.153$.

23. $a = 36992$, $b = 4021$.

24. $a = 410.5$, $c = 456.7$.

25. $A = 55° 44.1'$, $c = 411.5$.

26. $A = 58° 40'$, $b = 1.0405$.

27. $b = .3245$, $c = .8828$.

28. $a = 264.9$, $b = 75.95$.

29. $a = 176.64$, $c = 213.65$.

30. $A = 30° 17.2'$, $c = 20.04$.

31. $c = 25.206$.

32. $a = 1.735$.

33. $c = 2725.6$.

34. $A = 47° 42.9'$.

35. $a = .8346$.

36. $a = 49.25$.

37. 14.106 in.

38. 78.12 ft.

39. $31° 47.1'$.

40. $36° 37.9'$.

41. 99.45 miles.

42. 11.371 in.

43. $19° 50'$.

44. 399.5 ft.

45. First, 25.22 miles; second, 30.07 miles.

46. 21.65 ft.

47. 14.487 in.; 15.682 in.

48. 453.7 ft.

49. 17.26 in.

50. 389.4 ft.

51. 437.6.

52. 5.773 in.

53. 481.9 ft.

54. Rate, 6.792 miles an hour; bearing, $63° 8.4'$ west of north.

Art. 112; page 74.

2. 69.03.

3. .151.

4. 5695.

5. .08938.

6. 8208.

7. .002245.

8. 1.223.

9. 107.2.

10. .1017.

Digitized by Google

Art. 121; page 84.

2. $b = 15.837$, $c = 14.703$.
3. $a = .445$, $c = .9942$.
4. $a = .01913$, $b = .02272$.
5. $a = 33.78$, $c = 18.54$.
6. $b = 8.24$, $c = 5.464$.
7. $a = .011162$, $b = .006962$.
8. $b = 447.4$, $c = 425.7$.
9. $a = 342.6$, $c = 303.3$.
10. $a = 11.067$, $b = 6.067$.

Art. 122; page 85.

2. $A = 100° 56.7'$, $b = 19.78$.
3. $A = 83° 14.7'$, $c = 383.5$.
4. $B = 39° 15.8'$, $a = 3.211$.
5. $A = 24° 41.8'$, $c = .5886$.
6. $B = 143° 29.1'$, $a = 886.4$.
7. $A = 35° 1.9'$, $b = 12.78$.
8. $A = 59° 16.3'$, $c = 60.74$.
9. $B = 21° 1.2'$, $a = .06699$.
10. $A = 40° 52' 19''$, $b = 77.14$.

Art. 123; page 88.

3. $A = 28° 57.4'$, $B = 46° 34.2'$, $C = 104° 28.4'$.
4. $A = 34° 46.4'$, $B = 86° 24.8'$, $C = 58° 49'$.
5. $A = 74° 40'$, $B = 47° 46.4'$, $C = 57° 33.4'$.
6. $A = 58° 26'$, $B = 74° 23.8'$, $C = 47° 10.6'$.
7. $A = 55° 55.4'$, $B = 79° 43.8'$, $C = 44° 20'$.
8. $A = 49° 24.2'$, $B = 58° 38'$, $C = 71° 57.8'$.
9. $A = 60° 50.8'$, $B = 46° 6.2'$, $C = 73° 1.6'$.
10. $A = 18° 12.4'$, $B = 135° 50.6'$, $C = 25° 56.6'$.

Art. 127; pages 91 and 92.

6. $B = 39° 21.3'$, $c = 5.511$.
7. $B = 33° 28.4'$, $a = 118.33$;
or, $B = 146° 31.6'$, $a = 14.58$.
8. $C = 53° 18.9'$, $a = .07508$.
9. $A = 19° 18.1'$, $b = 2.522$.

Digitized by Google

10. $A = 79° 20'$, $c = .11416$;
or, $A = 100° 40'$, $c = .05121$.

11. Impossible.

12. $B = 24° 5.4'$, $a = 103.3$.

13. $C = 90°$, $b = 5.007$.

14. $C = 67° 10'$, $a = 6.918$;
or, $C = 112° 50'$, $a = 2.913$.

15. $A = 46° 53.3'$, $c = 141.48$.

16. Impossible.

17. $C = 24° 31.4'$, $b = 1.0637$.

18. $B = 90°$, $c = 127.32$.

19. $A = 70° 12'$, $b = .2879$;
or, $A = 109° 48'$, $b = .1045$.

20. $C = 37° 10'$, $a = 289.3$.

Art. 128; page 93.

2. 1077.9.
3. 14.697.
4. 3.257.
5. 5114.
6. 2604.
7. 1.353.
8. .06858.
9. 215.9.
10. .1273.
11. 4491.
12. .00001817.
13. 11.732.

Art. 129; pages 93 and 94.

1. Height of tower, 153.64 ft.; distance from first point, 117.27 ft.; from second, 217.27 ft.

2. 29800 square rods.

3. 247.7 ft.

4. 4.927 miles.

5. From first position, 9.9 miles; from second, 19.122 miles.

6. 298 ft.

7. Distance, 91.66 ft.; height, 33.9 ft.

8. 1113.1 ft.
9. 248 ft.
10. 1569.6.

Digitized by Google

Art. 153; page 112.

5. $A = 148° 5'$, $B = 65° 23.2'$, $b = 38° 2'$.
6. $a = 40° 41.8'$, $b = 134° 30.8'$, $c = 122° 7.5'$.
7. $A = 149° 41'$, $B = 66° 43.8'$, $c = 137° 20'$.
8. $A = 27° 11.6'$, $a = 25° 25.2'$, $c = 69° 54'$;
or, $A = 152° 48.4'$, $a = 154° 34.8'$, $c = 110° 6'$.
9. $A = 109° 22.5'$, $a = 110° 57.6'$, $b = 113° 22'$.
10. $A = 66° 12.1'$, $b = 127° 17.4'$, $c = 107° 5.1'$.
11. $A = 72° 28.9'$, $B = 140° 38.1'$, $c = 112° 37.7'$.
12. $B = 27° 7.2'$, $b = 25° 24.4'$, $c = 109° 46'$;
or, $B = 152° 52.8'$, $b = 154° 35.6'$, $c = 70° 14'$.
13. $A = 120° 44.3'$, $a = 156° 30'$, $B = 33° 52.6'$.
14. $a = 41° 5.5'$, $b = 26° 25'$, $c = 47° 32.1'$.
15. $a = 60° 31.4'$, $B = 143° 50'$, $b = 147° 32.1'$.
16. $A = 78° 46.7'$, $a = 70° 10'$, $c = 106° 27.5'$;
or, $A = 101° 13.3'$, $a = 109° 50'$, $c = 73° 32.5'$.
17. $A = 30° 32.1'$, $a = 22° 1.1'$, $b = 43° 17.9'$.
18. $a = 166° 8.7'$, $B = 101° 48.9'$, $c = 50° 18.4'$.
19. $A = 112° 2.5'$, $B = 109° 12'$, $c = 81° 53.6'$.
20. $A = 135° 2.5'$, $b = 68° 17.3'$, $c = 105° 44'$.
21. $a = 146° 32.5'$, $b = 109° 48.1'$, $c = 73° 35'$.
22. $A = 70°$, $B = 75° 6.2'$, $b = 74° 7.3'$.

Art. 154; page 113.

2. $a = 117° 1.2'$, $B = 153° 41.9'$, $C = 132° 34'$.
3. $a = 57° 22.1'$, $b = 129° 41.6'$, $C = 57° 52.5'$.
4. $A = 20° 0.9'$, $B = 141° 29.6'$, $b = 113° 17.1'$.
5. $A = 33° 27.5'$, $a = 35° 4.4'$, $b = 78° 46.7'$.
6. $B = 69° 16'$, $b = 70°$, $C = 84° 30'$;
or, $B = 110° 44'$, $b = 110°$, $C = 95° 30'$.

Digitized by Google

Art. 167; page 125.

2. $a = 95° 37.8'$, $b = 41° 52.2'$, $C = 110° 48.8'$.
3. $b = 120° 16.6'$, $c = 69° 19.6'$, $A = 50° 26.2'$.
4. $a = 34° 3'$, $c = 64° 19.4'$, $B = 37° 39.6'$.
5. $a = 69° 4'$, $b = 146° 25.6'$, $C = 125° 12.2'$.

Art. 168; page 126.

2. $A = 121° 32.8'$, $B = 40° 56.8'$, $c = 37° 25.8'$.
3. $A = 86° 59.7'$, $C = 60° 50.9'$, $b = 111° 17'$.
4. $B = 134° 57.3'$, $C = 50° 41.1'$, $a = 69° 8.8'$.
5. $A = 147° 29'$, $B = 163° 8.6'$, $c = 76° 8.4'$.

Art. 169; page 127.

2. $A = 51° 58.2'$, $B = 83° 54.4'$, $C = 58° 53.2'$.
3. $A = 142° 32.8'$, $B = 27° 52.6'$, $C = 32° 27.2'$.
4. $A = 35° 31'$, $B = 24° 42.6'$, $C = 138° 24.8'$.
5. $A = 47° 21.2'$, $B = 42° 19.4'$, $C = 124° 38'$.

Art. 170; page 128.

2. $a = 67° 51.8'$, $b = 71° 44.4'$, $c = 57°$.
3. $a = 144° 9.8'$, $b = 148° 48.6'$, $c = 41° 43.6'$.
4. $a = 89° 51.2'$, $b = 85° 49.2'$, $c = 72° 31.8'$.
5. $a = 100° 4.6'$, $b = 49° 58.8'$, $c = 60° 6'$.

Art. 171; page 131.

4. $C = 65° 30'$, $A = 97° 20'$, $a = 100° 44.6'$.
5. $B = 42° 40'$, $C = 159° 54'$, $c = 153° 30.2'$;
or, $B = 137° 20'$, $C = 50° 20.6'$, $c = 90° 9.6'$.
6. Impossible.
7. $C = 90°$, $B = 113° 36.9'$, $b = 114° 51.9'$.

Digitized by Google

8. $B = 68° 18'$, $A = 132° 33.8'$, $a = 131° 15.8'$;
or, $B = 111° 42'$, $A = 77° 4.6'$, $a = 95° 50'$.

9. Impossible.

10. $C = 146° 37.9'$, $B = 55° 0.6'$, $b = 96° 34.4'$.

Art. 172; page 133.

2. $b = 114° 50'$, $A = 79° 20'$, $a = 82° 56'$.

3. $a = 67° 24'$, $C = 164° 6.4'$, $c = 160° 6.4'$;
or, $a = 112° 36'$, $C = 128° 20.6'$, $c = 103° 2.4'$.

4. $c = 90°$, $B = 63° 42.7'$, $b = 66° 26.2'$.

5. Impossible.

6. $b = 27° 22.1'$, $A = 47° 21.2'$, $a = 117° 9.2'$.

7. $a = 43° 3.1'$, $B = 89° 23.8'$, $b = 129° 8.4'$;
or, $a = 136° 56.9'$, $B = 26° 58.6'$, $b = 20° 35.8'$.

8. Impossible.

Art. 173; pages 134 and 136.

1. Distance, 3275 miles; bearing of Greenwich from Boston, N. 53° 7′ E.; of Boston from Greenwich, N. 71° 39.4′ W.

2. Distance, 11010 miles; bearing of Calcutta from Valparaiso, S. 64° 20.4′ E.; of Valparaiso from Calcutta, S. 54° 54.6′ W.

3. Latitude 49° 58.4′ N.

4. 6 h., 0 m., 40 s., A.M.; longitude 44° 50′ W.

5. 15° 0.8′.

6. 8 h., 2 m., 40 s., A.M.

7. 5 h., 3 m., 26.4 s., A.M.

Digitized by Google

Digitized by Google

TABLES

OF THE

LOGARITHMS OF NUMBERS FROM 100 TO 1000,

AND OF THE

LOGARITHMIC AND NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS OF ANGLES AT INTERVALS OF TEN MINUTES FROM 0° TO 90°,

CALCULATED TO FOUR PLACES OF DECIMALS.

BY

WEBSTER WELLS, S.B.,

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

LEACH, SHEWELL, AND SANBORN.

BOSTON. NEW YORK. CHICAGO.

Digitized by Google

COPYRIGHT, 1887,
BY WEBSTER WELLS.

TYPOGRAPHY BY
J. S. CUSHING & CO

PRESSWORK BY
BERWICK & SMITH.

Digitized by Google

USE OF THE TABLES.

I. USE OF THE TABLE OF LOGARITHMS OF NUMBERS.

This table (pages 12 and 13) gives the mantissæ of the logarithms of all numbers of three figures from 100 to 1000, calculated to four places of decimals.

To find the Logarithm of any Number of Three Figures.

Find in the column headed "No." the first two figures of the given number. Then the mantissa required will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic by the rules of Art. 86.

For example, $\log 168 = 2.2253$;

$\log .0344 = 8.5366 - 10$; etc.

For numbers of one or two figures, the column headed 0 may be used; for by Art. 97, log 83 has the same mantissa as log 830, and log 9 the same mantissa as log 900.

Thus, $\log 83 = 1.9191$, and $\log 9 = 0.9542$.

To find the Logarithm of a Number of more than Three Figures.

1. Required the logarithm of 327.6.

We find from the table,

$$\log 327 = 2.5145,$$
$$\log 328 = 2.5159.$$

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Digitized by Google

Therefore an increase of .6 of a unit in the number will produce an increase of $.6 \times .0014$ in the logarithm, or .0008 to the nearest fourth decimal place.

Hence, $\log 327.6 = 2.5145 + .0008 = 2.5153$.

Note I. The difference between any mantissa in the table and the mantissa of the next higher number of three figures, is called the *tabular difference*. The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

EXAMPLES.

2. Find the logarithm of .021508.

Mantissa of 215 = 3324 Tabular difference = 21

2 .08

3326 Correction = 1.68

= 2, nearly.

Result, $8.3326 - 10$.

Find the logarithms of the following:

3. 80.	**7.** .7723.	**11.** 20.08.	**15.** 5.1809.
4. 6.3.	**8.** 1056.	**12.** 92461.	**16.** 1036.5.
5. 298.	**9.** 3.294.	**13.** .40322.	**17.** .086676.
6. .902.	**10.** .05205.	**14.** .007178.	**18.** .11507.

To find the Number Corresponding to a Logarithm.

1. Required the number whose logarithm is 1.6571.

Digitized by Google

Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two figures to the left of the decimal point (Art. 86). Hence,

Number corresponding to $1.6571 = 45.4$.

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissa 3927, whose corresponding number is 247, and the mantissa 3945, whose corresponding number is 248.

That is, an increase of 18 in the mantissa produces an increase of one unit in the corresponding number.

Hence, an increase of 7 in the mantissa will produce an increase of $\frac{7}{18}$ of a unit in the number, or .39 nearly; that is,

Number corresponding $= 247 + .39 = 247.39$.

We derive the following rule from the above operation:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

EXAMPLES.

3. Find the number whose logarithm is $7.0256 - 10$.

0256

Next less mantissa $= 0253$; three figures corresponding $= 106$.

Tabular difference $= 41)3.00(.07$, nearly.

Since the characteristic is $7 - 10$, there must be two ciphers between the decimal point and first significant figure (Art. 86).

Hence, Number corresponding $= .0010607$.

Digitized by Google

Find the numbers corresponding to the following logarithms:

4. 1.8055.	**9.** 8.1648 − 10.	**14.** 1.6482.
5. 9.4487 − 10.	**10.** 7.5209 − 10.	**15.** 7.0450 − 10.
6. 0.2165.	**11.** 4.0095.	**16.** 4.8016.
7. 3.9487.	**12.** 0.9774.	**17.** 8.1144 − 10.
8. 2.7364.	**13.** 9.3178 − 10.	**18.** 2.7015.

II. USE OF THE TABLE OF LOGARITHMIC SINES, COSINES, ETC.

This table (pages 14 to 19) gives the logarithms of the sines, cosines, tangents, and cotangents of all angles at intervals of 10 minutes from 0° to 90°.

For angles between 0° and 45°, the degrees and minutes will be found in the *left-hand* column, and the functions in the columns designated by the names at the *top;* that is, sines in the first column, cosines in the second, tangents in the third, and cotangents in the fourth.

For angles between 45° and 90°, the degrees and minutes will be found in the *right-hand* column, and the functions in the columns designated by the names at the *foot;* that is, cosines in the first column, sines in the second, cotangents in the third, and tangents in the fourth.

If only the *mantissa* of the logarithm is found, the characteristic may be obtained from the nearest logarithm in the same column, above or below according as the angle is below or above 45°, in which the characteristic is given.

Since the sines and cosines of all acute angles, the tangents of angles between 0° and 45°, and the cotangents of angles between 45° and 90°, are less than unity, the characteristics of their logarithms have been increased by 10, and − 10 must be written after the mantissa; in all other cases, the true value of the characteristic is given in the table.

Digitized by Google

Thus, $\log \sin 38° 30' = 9.7941 - 10$;

$\log \tan 66° 20' = 0.3583$;

$\log \cot 79° 10' = 9.2819 - 10$;

$\log \cos 89° 40' = 7.7648 - 10$; etc.

To find the Logarithmic Sine, Cosine, Tangent, or Cotangent, of any Acute Angle expressed in Degrees and Minutes.

RULE.

Find from the table the logarithmic sine, cosine, tangent, or cotangent of the next less multiple of ten minutes, and the difference for 1′ *corresponding.* (See Note II. below.)

Multiply this difference by the number of minutes remaining.

If sine or tangent, add
If cosine or cotangent, subtract } *this correction.*

Note II. The columns to the right of those headed "sin.," "cos.," and "tan.," contain the respective differences for 1′; the right-hand column of differences is also to be used with the column headed "cot."

It will be observed that the difference does not stand in the same horizontal line with the logarithm, but midway between it and the next. The difference should always be taken corresponding to the interval in which the angle is situated; thus, in finding log sin 79° 53′, the difference corresponding to the interval between 79° 50′ and 80° is 7.1.

Note III. The rule assumes that the differences of the logarithmic functions are proportional to the differences of their corresponding angles, which, unless the angle is very near to 0° or 90°, is in general sufficiently exact for practical purposes. (See page 8.)

Note IV. If the angle is expressed in degrees, minutes, and seconds, the seconds should be reduced to the decimal part of a minute before applying the rule.

EXAMPLES.

1. Find log tan 17° 14′.

$\log \tan 17° 10' = 9.4898 - 10$	D. 1′ = 4.5
18	4
Result, $9.4916 - 10$.	Corr. = 18.0

Digitized by Google

2. Find log cos 66° 33.5′.

log cos 66° 30′ =	9.6007 − 10	D. 1′ =	2.9
	10		3.5
Result,	9.5997 − 10.		1.45
			8.7
		Corr. =	10.15 = 10, nearly.

Find the values of the following:

3. log tan 35° 19′.

4. log sin 61° 58′.

5. log cot 12° 34′.

6. log cos 26° 56′.

7. log tan 82° 3′ 20″.

8. log sin 55° 11.8′.

9. log cos 30° 42.5′.

10. log cot 48° 3′ 43″.

TO FIND THE ACUTE ANGLE CORRESPONDING TO A GIVEN LOGARITHMIC SINE, COSINE, TANGENT, OR COTANGENT.

RULE.

Find in the table the next less logarithmic sine, cosine, tangent, or cotangent, the degrees and minutes corresponding, and the difference for 1′. (See Note II.)

Subtract the next less from the given logarithm, and divide the remainder by the difference for 1′, *giving the correction in minutes.*

If sine or tangent, add
If cosine or cotangent, subtract } *this correction.*

Note V. In searching for the next less logarithm, attention must be paid to the fact that the functions are found in different columns according as the angle is below or above 45°.

If, for example, the next less logarithmic sine is found in the column with "sin." at the *top*, the degrees and minutes must be taken from the *left-hand* column; but if it is found in the column with "sin." at the *foot*, they must be taken from the *right-hand* column. Similar considerations apply to the other three functions.

Note VI. It would of course be equally correct to look out the next *greater* logarithm, and *subtract* the correction for sines and tangents, and *add* it for cosines and cotangents.

Digitized by Google

EXAMPLES.

1. Find the angle whose log sin = 9.9594 − 10.

9.9594 − 10

Next less log sin = 9.9590 − 10 ; angle corresponding = 65° 30′.

D. 1′ = .6)4.00(6.7, nearly.

3.6

.40

Adding the correction, the angle corresponding is 65° 36.7′.

2. Find the angle whose log cot = 0.1696.

0.1696

Next less log cot = 0.1683 ; angle corresponding = 34° 10′.

D. 1′ = 2.7)13.00(4.8, nearly.

10.8

2.20

Subtracting the correction, the angle corresponding is 34° 5.2′.

Find the angles corresponding in the following cases:

3. log tan = 0.9164.
4. log cos = 9.9238 − 10.
5. log sin = 9.8630 − 10.
6. log cot = 0.2154.
7. log cos = 9.2279 − 10.
8. log cot = 9.4700 − 10.
9. log sin = 9.1891 − 10.
10. log tan = 0.0502.

To find the Logarithmic Secant or Cosecant of any Acute Angle.

Since $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$, we have by Art. 103,

$$\log \sec x = \operatorname{colog} \cos x, \text{ and } \log \csc x = \operatorname{colog} \sin x.$$

That is,

To find the logarithmic secant, subtract the logarithmic cosine from 10 − 10; *and to find the logarithmic cosecant, subtract the logarithmic sine from* 10 − 10.

Digitized by Google

EXAMPLES.

1. Find log sec 22° 38′.

From the table, we find

$$\log \cos 22° 38' = 9.9652 - 10.$$

Subtracting from $10 - 10$, we have

$$\log \sec 22° 38' = 0.0348.$$

Find the values of the following:

2. log csc 65° 12′. **3.** log sec 80° 7.3′.

Find the angles corresponding in the following cases:

4. log sec = 0.4003. **5.** log csc = 0.1888.

Note VII. The logarithmic cotangent of an angle may be obtained by subtracting the logarithmic tangent from $10 - 10$.

More Accurate Method for Angles near to 0° or 90°.

It was stated in Note III., page 5, that in general the differences of the logarithmic functions are approximately proportional to the differences of their corresponding angles. It will be seen from the table that this is not the case with the logarithmic sines, tangents, and cotangents of angles near to 0°, nor with the logarithmic cosines, tangents, and cotangents of angles near to 90°.

Thus, the tabular difference for 1′ in the case of the logarithmic sine, tangent, or cotangent of an angle between 40′ and 50′ is 96.9, while for an angle between 50′ and 1° it is 79.2.

A very accurate method for finding the logarithmic sine or tangent of an angle near to 0°, or the logarithmic cosine or cotangent of an angle near to 90°, is to first calculate the *natural function* by aid of the table of natural sines and cosines, or of natural tangents and cotangents, and then find the logarithm of the result.

Digitized by Google

To find the angle corresponding in similar cases, find the *number corresponding* to the logarithmic function, and then, by aid of the tables of natural functions, calculate the angle corresponding to the result.

EXAMPLES.

1. Required log sin 0° 56′.

From the table of natural sines and cosines, we obtain

$$\text{natural sin } 0° 56' = .016289.$$

Whence, $$\log \sin 0° 56' = 8.2119 - 10.$$

This result is correct to the last place of decimals; by the ordinary method we should have obtained 8.2102 − 10.

2. Required the angle whose log tan = 8.0302 − 10.

The number corresponding to this logarithm is .01072.

From the table of natural tangents and cotangents, the angle whose natural tangent is .01072 is 36.85′, or 36′ 51″.

The correct result is 36′ 51.1″; by the ordinary method we should have obtained 37.15′, or 37′ 9″.

Note VIII. To find with accuracy the log cotangent of an angle near to 0°, find the log tangent of the angle by the above method, and then take the *cologarithm* of the result. (See Note VII., page 8.)

To find the angle corresponding to a log cotangent in a similar case, find the log tangent of the angle (Note VII.), and then find the angle corresponding as above.

Note IX. To find the log tangent of an angle near to 90°, find the log tangent of its *complement*, and take the *cologarithm* of the result. (See Note VIII.)

To find the angle corresponding in a similar case, find the angle corresponding to its cologarithm, and take the complement of the result.

Note X. The more accurate method should be employed in finding the log sines, tangents, or cotangents of angles between 0° and 5°, or the log cosines, tangents, or cotangents of angles between 85° and 90°, and the angles corresponding in similar cases. For angles between 5° and 85° the ordinary method is sufficiently exact.

Digitized by Google

Find the values of the following:

3. log sin 3° 32′.

4. log cos 88° 17′.

5. log tan 2° 28.2′.

6. log cot 55′ 14″.

Find the angles corresponding in the following cases:

7. log sin = 7.8702 − 10.

8. log cos = 8.6150 − 10.

9. log cot = 8.2892 − 10.

10. log tan = 1.7869.

To find the Logarithmic Functions of any Angle not lying between the Limits 0° and 90°.

By Art. 54, any function of any angle may be expressed as a function of a certain acute angle; and hence the table of the functions of acute angles serves to determine the functions of angles of any magnitude whatever, positive or negative.

For example, required log sin 152° 16′.

By Art. 54, $\sin 152° 16' = \sin(90° + 62° 16') = \cos 62° 16'$;

whence, $\log \sin 152° 16' = \log \cos 62° 16' = 9.6678 - 10$.

Or we may proceed as follows:

$$\sin 152° 16' = \sin(180° - 27° 44') = \sin 27° 44'.$$

Note XI. If the natural function is *negative*, as for example in the case of the cosine of an angle between 90° and 180°, there is no logarithmic function, strictly speaking. (See Note to Art. 105.)

In the solution of examples involving such functions, we may proceed as if the functions were positive, and determine the algebraic sign of the result irrespective of the logarithmic work. Illustrations of this will be found in Chapters XI. and XII.

III. USE OF THE TABLE OF NATURAL SINES AND COSINES.

This table (pages 20 and 21) gives the natural values of the sines and cosines of all angles at intervals of 10 minutes from 0° to 90°, calculated to four places of decimals.

Digitized by Google

Its use is similar to that of the table of logarithmic functions, except that the tabular differences for 1′ are not given, but are to be calculated from the table when required.

EXAMPLES.

1. Required sin 48° 52′.

$$\sin 48° 50' = .7528.$$

The difference between this and sin 49° 0′ is .0019.

$$\text{Difference for } 1' = .00019$$
$$\underline{\qquad\qquad 2}$$
$$\text{Correction for } 2' = .00038 = .0004, \text{ nearly.}$$

Therefore $\sin 48° 52' = .7528 + .0004 = .7532.$

2. Required the angle whose cos = .6103.

$$\text{Next less cos} = \begin{array}{r} .6103 \\ \underline{.6088} \end{array}; \text{ angle corresponding} = 52° 30'.$$

$$\text{Difference for } 1' = 2.3)15.00(6.5, \text{ nearly.}$$
$$\underline{\quad 13.8}$$
$$1.20$$

Subtracting the correction, angle corresponding = 52° 23.5′.

Find the values of the following:

3. sin 17° 33′. **4.** cos 75° 48.3′.

Find the angles corresponding in the following cases:

5. sin = .7385. **6.** cos = .9280.

IV. USE OF THE TABLE OF NATURAL TANGENTS AND COTANGENTS.

This table (pages 22 and 23) gives the tangents and cotangents of all angles at intervals of 10 minutes from 0° to 90°; its use is similar to that of the table of natural sines and cosines.

Digitized by Google

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

Digitized by Google

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Digitized by Google

Angle.	Sin.	D. 1′.	Cos.	D. 1′.	Tan.	D. 1′.	Cot.	
0° 0′	−∞		10.0000	.0	−∞		∞	90° 0′
0° 10′	7.4637	301.1	.0000	.0	7.4637	301.1	2.5363	89° 50′
0° 20′	.7648	176.0	.0000	.0	.7648	176.1	.2352	89° 40′
0° 30′	.9408	125.0	.0000	.0	.9409	124.9	.0591	89° 30′
0° 40′	8.0658	96.9	.0000	.0	8.0658	96.9	1.9342	89° 20′
0° 50′	.1627	79.2	.0000	.1	.1627	79.2	.8373	89° 10′
1° 0′	8.2419	66.9	9.9999	.0	8.2419	67.0	1.7581	89° 0′
1° 10′	.3088	58.0	.9999	.0	.3089	58.0	.6911	88° 50′
1° 20′	.3668	51.1	.9999	.0	.3669	51.2	.6331	88° 40′
1° 30′	.4179	45.8	.9999	.1	.4181	45.7	.5819	88° 30′
1° 40′	.4637	41.3	.9998	.0	.4638	41.5	.5362	88° 20′
1° 50′	.5050	37.8	.9998	.1	.5053	37.8	.4947	88° 10′
2° 0′	8.5428	34.8	9.9997	.0	8.5431	34.8	1.4569	88° 0′
2° 10′	.5776	32.1	.9997	.1	.5779	32.2	.4221	87° 50′
2° 20′	.6097	30.0	.9996	.0	.6101	30.0	.3899	87° 40′
2° 30′	.6397	28.0	.9996	.1	.6401	28.1	.3599	87° 30′
2° 40′	.6677	26.3	.9995	.0	.6682	26.3	.3318	87° 20′
2° 50′	.6940	24.8	.9995	.1	.6945	24.9	.3055	87° 10′
3° 0′	8.7188	23.5	9.9994	.1	8.7194	23.5	1.2806	87° 0′
3° 10′	.7423	22.2	.9993	.0	.7429	22.3	.2571	86° 50′
3° 20′	.7645	21.2	.9993	.1	.7652	21.3	.2348	86° 40′
3° 30′	.7857	20.2	.9992	.1	.7865	20.2	.2135	86° 30′
3° 40′	.8059	19.2	.9991	.1	.8067	19.4	.1933	86° 20′
3° 50′	.8251	18.5	.9990	.1	.8261	18.5	.1739	86° 10′
4° 0′	8.8436	17.7	9.9989	.0	8.8446	17.8	1.1554	86° 0′
4° 10′	.8613	17.0	.9989	.1	.8624	17.1	.1376	85° 50′
4° 20′	.8783	16.3	.9988	.1	.8795	16.5	.1205	85° 40′
4° 30′	.8946	15.8	.9987	.1	.8960	15.8	.1040	85° 30′
4° 40′	.9104	15.2	.9986	.1	.9118	15.4	.0882	85° 20′
4° 50′	.9256	14.7	.9985	.2	.9272	14.8	.0728	85° 10′
5° 0′	8.9403	14.2	9.9983	.1	8.9420	14.3	1.0580	85° 0′
5° 10′	.9545	13.7	.9982	.1	.9563	13.8	.0437	84° 50′
5° 20′	.9682	13.4	.9981	.1	.9701	13.5	.0299	84° 40′
5° 30′	.9816	12.9	.9980	.1	.9836	13.0	.0164	84° 30′
5° 40′	.9945	12.5	.9979	.2	.9966	12.7	.0034	84° 20′
5° 50′	9.0070	12.2	.9977	.1	9.0093	12.3	0.9907	84° 10′
6° 0′	9.0192	11.9	9.9976	.1	9.0216	12.0	0.9784	84° 0′
6° 10′	.0311	11.5	.9975	.2	.0336	11.7	.9664	83° 50′
6° 20′	.0426	11.3	.9973	.1	.0453	11.4	.9547	83° 40′
6° 30′	.0539	10.9	.9972	.1	.0567	11.1	.9433	83° 30′
6° 40′	.0648	10.7	.9971	.2	.0678	10.8	.9322	83° 20′
6° 50′	.0755	10.4	.9969	.1	.0786	10.5	.9214	83° 10′
7° 0′	9.0859	10.2	9.9968	.2	9.0891	10.4	0.9109	83° 0′
7° 10′	.0961	9.9	.9966	.2	.0995	10.1	.9005	82° 50′
7° 20′	.1060	9.7	.9964	.1	.1096	9.8	.8904	82° 40′
7° 30′	.1157		.9963		.1194		.8806	82° 30′
	Cos.	D. 1′.	Sin.	D. 1′.	Cot.	D. 1′.	Tan.	Angle.

Digitized by Google

Angle.	Sin.	D. 1′.	Cos.	D. 1′.	Tan.	D. 1′.	Cot.	
7° 30′	9.1157		9.9963	.2	9.1194		0.8806	82° 30′
7° 40′	.1252	9.5	.9961	.2	.1291	9.7	.8709	82° 20′
7° 50′	.1345	9.3	.9959	.1	.1385	9.4	.8615	82° 10′
8° 0′	9.1436	9.1	9.9958	.2	9.1478	9.3	0.8522	**82° 0′**
8° 10′	.1525	8.9	.9956	.2	.1569	9.1	.8431	81° 50′
8° 20′	.1612	8.7	.9954	.2	.1658	8.9	.8342	81° 40′
8° 30′	.1697	8.5	.9952	.2	.1745	8.7	.8255	81° 30′
8° 40′	.1781	8.4	.9950	.2	.1831	8.6	.8169	81° 20′
8° 50′	.1863	8.2	.9948	.2	.1915	8.4	.8085	81° 10′
9° 0′	9.1943	8.0	9.9946	.2	9.1997	8.2	0.8003	**81° 0′**
9° 10′	.2022	7.9	.9944	.2	.2078	8.1	.7922	80° 50′
9° 20′	.2100	7.8	.9942	.2	.2158	8.0	.7842	80° 40′
9° 30′	.2176	7.6	.9940	.2	.2236	7.8	.7764	80° 30′
9° 40′	.2251	7.5	.9938	.2	.2313	7.7	.7687	80° 20′
9° 50′	.2324	7.3	.9936	.2	.2389	7.6	.7611	80° 10′
10° 0′	9.2397	7.3	9.9934	.3	9.2463	7.4	0.7537	**80° 0′**
10° 10′	.2468	7.1	.9931	.2	.2536	7.3	.7464	79° 50′
10° 20′	.2538	7.0	.9929	.2	.2609	7.3	.7391	79° 40′
10° 30′	.2606	6.8	.9927	.2	.2680	7.1	.7320	79° 30′
10° 40′	.2674	6.8	.9924	.3	.2750	7.0	.7250	79° 20′
10° 50′	.2740	6.6	.9922	.2	.2819	6.9	.7181	79° 10′
11° 0′	9.2806	6.6	9.9919	.3	9.2887	6.8	0.7113	**79° 0′**
11° 10′	.2870	6.4	.9917	.2	.2953	6.6	.7047	78° 50′
11° 20′	.2934	6.4	.9914	.3	.3020	6.7	.6980	78° 40′
11° 30′	.2997	6.3	.9912	.2	.3085	6.5	.6915	78° 30′
11° 40′	.3058	6.1	.9909	.3	.3149	6.4	.6851	78° 20′
11° 50′	.3119	6.1	.9907	.2	.3212	6.3	.6788	78° 10′
12° 0′	9.3179	6.0	9.9904	.3	9.3275	6.3	0.6725	**78° 0′**
12° 10′	.3238	5.9	.9901	.3	.3336	6.1	.6664	77° 50′
12° 20′	.3296	5.8	.9899	.2	.3397	6.1	.6603	77° 40′
12° 30′	.3353	5.7	.9896	.3	.3458	6.1	.6542	77° 30′
12° 40′	.3410	5.7	.9893	.3	.3517	5.9	.6483	77° 20′
12° 50′	.3466	5.6	.9890	.3	.3576	5.9	.6424	77° 10′
13° 0′	9.3521	5.5	9.9887	.3	9.3634	5.8	0.6366	**77° 0′**
13° 10′	.3575	5.4	.9884	.3	.3691	5.7	.6309	76° 50′
13° 20′	.3629	5.4	.9881	.3	.3748	5.7	.6252	76° 40′
13° 30′	.3682	5.3	.9878	.3	.3804	5.6	.6196	76° 30′
13° 40′	.3734	5.2	.9875	.3	.3859	5.5	.6141	76° 20′
13° 50′	.3786	5.2	.9872	.3	.3914	5.5	.6086	76° 10′
14° 0′	9.3837	5.1	9.9869	.3	9.3968	5.4	0.6032	**76° 0′**
14° 10′	.3887	5.0	.9866	.3	.4021	5.3	.5979	75° 50′
14° 20′	.3937	5.0	.9863	.3	.4074	5.3	.5926	75° 40′
14° 30′	.3986	4.9	.9859	.4	.4127	5.3	.5873	75° 30′
14° 40′	.4035	4.9	.9856	.3	.4178	5.1	.5822	75° 20′
14° 50′	.4083	4.8	.9853	.3	.4230	5.2	.5770	75° 10′
15° 0′	9.4130	4.7	9.9849	.4	9.4281	5.1	0.5719	**75° 0′**
	Cos.	D. 1′.	Sin.	D. 1′.	Cot.	D. 1′.	Tan.	Angle.

Digitized by Google

Angle.	Sin.	D. 1′.	Cos.	D. 1′.	Tan.	D. 1′.	Cot.	
15° 0′	9.4130	4.7	9.9849	.3	9.4281	5.0	0.5719	**75° 0′**
15° 10′	.4177	4.6	.9846	.3	.4331	5.0	.5669	74° 50′
15° 20′	.4223	4.6	.9843	.3	.4381	4.9	.5619	74° 40′
15° 30′	.4269	4.5	.9839	.4	.4430	4.9	.5570	74° 30′
15° 40′	.4314	4.5	.9836	.3	.4479	4.8	.5521	74° 20′
15° 50′	.4359	4.4	.9832	.4	.4527	4.8	.5473	74° 10′
16° 0′	9.4403	4.4	9.9828	.4	9.4575	4.7	0.5425	**74° 0′**
16° 10′	.4447	4.4	.9825	.3	.4622	4.7	.5378	73° 50′
16° 20′	.4491	4.2	.9821	.4	.4669	4.7	.5331	73° 40′
16° 30′	.4533	4.3	.9817	.4	.4716	4.6	.5284	73° 30′
16° 40′	.4576	4.2	.9814	.3	.4762	4.6	.5238	73° 20′
16° 50′	.4618	4.1	.9810	.4	.4808	4.5	.5192	73° 10′
17° 0′	9.4659	4.1	9.9806	.4	9.4853	4.5	0.5147	**73° 0′**
17° 10′	.4700	4.1	.9802	.4	.4898	4.5	.5102	72° 50′
17° 20′	.4741	4.0	.9798	.4	.4943	4.4	.5057	72° 40′
17° 30′	.4781	4.0	.9794	.4	.4987	4.4	.5013	72° 30′
17° 40′	.4821	4.0	.9790	.4	.5031	4.4	.4969	72° 20′
17° 50′	.4861	3.9	.9786	.4	.5075	4.3	.4925	72° 10′
18° 0′	9.4900	3.9	9.9782	.4	9.5118	4.3	0.4882	**72° 0′**
18° 10′	.4939	3.8	.9778	.4	.5161	4.2	.4839	71° 50′
18° 20′	.4977	3.8	.9774	.4	.5203	4.2	.4797	71° 40′
18° 30′	.5015	3.7	.9770	.5	.5245	4.2	.4755	71° 30′
18° 40′	.5052	3.8	.9765	.4	.5287	4.2	.4713	71° 20′
18° 50′	.5090	3.6	.9761	.4	.5329	4.1	.4671	71° 10′
19° 0′	9.5126	3.7	9.9757	.5	9.5370	4.1	0.4630	**71° 0′**
19° 10′	.5163	3.6	.9752	.4	.5411	4.0	.4589	70° 50′
19° 20′	.5199	3.6	.9748	.5	.5451	4.0	.4549	70° 40′
19° 30′	.5235	3.5	.9743	.4	.5491	4.0	.4509	70° 30′
19° 40′	.5270	3.6	.9739	.5	.5531	4.0	.4469	70° 20′
19° 50′	.5306	3.5	.9734	.4	.5571	4.0	.4429	70° 10′
20° 0′	9.5341	3.4	9.9730	.5	9.5611	3.9	0.4389	**70° 0′**
20° 10′	.5375	3.4	.9725	.4	.5650	3.9	.4350	69° 50′
20° 20′	.5409	3.4	.9721	.5	.5689	3.8	.4311	69° 40′
20° 30′	.5443	3.4	.9716	.5	.5727	3.9	.4273	69° 30′
20° 40′	.5477	3.3	.9711	.5	.5766	3.8	.4234	69° 20′
20° 50′	.5510	3.3	.9706	.5	.5804	3.8	.4196	69° 10′
21° 0	9.5543	3.3	9.9702	.4	9.5842	3.7	0.4158	**69° 0′**
21° 10′	.5576	3.3	.9697	.5	.5879	3.8	.4121	68° 50′
21° 20′	.5609	3.2	.9692	.5	.5917	3.7	.4083	68° 40′
21° 30′	.5641	3.2	.9687	.5	.5954	3.7	.4046	68° 30′
21° 40′	.5673	3.1	.9682	.5	.5991	3.7	.4009	68° 20′
21° 50′	.5704	3.2	.9677	.5	.6028	3.6	.3972	68° 10′
22° 0′	9.5736	3.1	9.9672	.5	9.6064	3.6	0.3936	**68° 0′**
22° 10′	.5767	3.1	.9667	.6	.6100	3.6	.3900	67° 50′
22° 20′	.5798	3.0	.9661	.5	.6136	3.6	.3864	67° 40′
22° 30′	.5828		.9656		.6172		.3828	67° 30′
	Cos.	**D. 1′.**	**Sin.**	**D. 1′.**	**Cot.**	**D. 1′.**	**Tan.**	**Angle.**

Digitized by Google

Angle.	Sin.	D. 1′.	Cos.	D. 1′.	Tan.	D. 1′.	Cot.	
22° 30′	9.5828	3.1	9.9656	.5	9.6172	3.6	0.3828	67° 30′
22° 40′	.5859	3.0	.9651	.5	.6208	3.5	.3792	67° 20′
22° 50′	.5889	3.0	.9646	.6	.6243	3.6	.3757	67° 10′
23° 0′	9.5919	2.9	9.9640	.5	9.6279	3.5	0.3721	**67° 0′**
23° 10′	.5948	3.0	.9635	.6	.6314	3.4	.3686	66° 50′
23° 20′	.5978	2.9	.9629	.5	.6348	3.5	.3652	66° 40′
23° 30′	.6007	2.9	.9624	.6	.6383	3.4	.3617	66° 30′
23° 40′	.6036	2.9	.9618	.5	.6417	3.5	.3583	66° 20′
23° 50′	.6065	2.8	.9613	.6	.6452	3.4	.3548	66° 10′
24° 0′	9.6093	2.8	9.9607	.5	9.6486	3.4	0.3514	**66° 0′**
24° 10′	.6121	2.8	.9602	.6	.6520	3.3	.3480	65° 50′
24° 20′	.6149	2.8	.9596	.6	.6553	3.4	.3447	65° 40′
24° 30′	.6177	2.8	.9590	.6	.6587	3.3	.3413	65° 30′
24° 40′	.6205	2.7	.9584	.5	.6620	3.4	.3380	65° 20′
24° 50′	.6232	2.7	.9579	.6	.6654	3.3	.3346	65° 10′
25° 0′	9.6259	2.7	9.9573	.6	9.6687	3.3	0.3313	**65° 0′**
25° 10′	.6286	2.7	.9567	.6	.6720	3.2	.3280	64° 50′
25° 20′	.6313	2.7	.9561	.6	.6752	3.3	.3248	64° 40′
25° 30′	.6340	2.6	.9555	.6	.6785	3.2	.3215	64° 30′
25° 40′	.6366	2.6	.9549	.6	.6817	3.3	.3183	64° 20′
25° 50′	.6392	2.6	.9543	.6	.6850	3.2	.3150	64° 10′
26° 0′	9.6418	2.6	9.9537	.7	9.6882	3.2	0.3118	**64° 0′**
26° 10′	.6444	2.6	.9530	.6	.6914	3.2	.3086	63° 50′
26° 20′	.6470	2.5	.9524	.6	.6946	3.1	.3054	63° 40′
26° 30′	.6495	2.6	.9518	.6	.6977	3.2	.3023	63° 30′
26° 40′	.6521	2.5	.9512	.7	.7009	3.1	.2991	63° 20′
26° 50′	.6546	2.4	.9505	.6	.7040	3.2	.2960	63° 10′
27° 0′	9.6570	2.5	9.9499	.7	9.7072	3.1	0.2928	**63° 0′**
27° 10′	.6595	2.5	.9492	.6	.7103	3.1	.2897	62° 50′
27° 20′	.6620	2.4	.9486	.7	.7134	3.1	.2866	62° 40′
27° 30′	.6644	2.4	.9479	.6	.7165	3.1	.2835	62° 30′
27° 40′	.6668	2.4	.9473	.7	.7196	3.0	.2804	62° 20′
27° 50′	.6692	2.4	.9466	.7	.7226	3.1	.2774	62° 10′
28° 0′	9.6716	2.4	9.9459	.6	9.7257	3.0	0.2743	**62° 0′**
28° 10′	.6740	2.3	.9453	.7	.7287	3.0	.2713	61° 50′
28° 20′	.6763	2.4	.9446	.7	.7317	3.1	.2683	61° 40′
28° 30′	.6787	2.3	.9439	.7	.7348	3.0	.2652	61° 30′
28° 40′	.6810	2.3	.9432	.7	.7378	3.0	.2622	61° 20′
28° 50′	.6833	2.3	.9425	.7	.7408	3.0	.2592	61° 10′
29° 0′	9.6856	2.2	9.9418	.7	9.7438	2.9	0.2562	**61 °0′**
29° 10′	.6878	2.3	.9411	.7	.7467	3.0	.2533	60° 50′
29° 20′	.6901	2.2	.9404	.7	.7497	2.9	.2503	60° 40′
29° 30′	.6923	2.3	.9397	.7	.7526	3.0	.2474	60° 30′
29° 40′	.6946	2.2	.9390	.7	.7556	2.9	.2444	60° 20′
29° 50′	.6968	2.2	.9383	.8	.7585	2.9	.2415	60° 10′
30° 0′	9.6990		9.9375		9.7614		0.2386	**60° 0′**
	Cos.	D. 1′.	Sin.	D. 1′.	Cot.	D. 1′.	Tan.	Angle.

Digitized by Google

Angle.	Sin.	D. 1′.	Cos.	D. 1′.	Tan.	D. 1′.	Cot.	
30° 0′	9.6990	2.2	9.9375	.7	9.7614	3.0	0.2386	**60° 0′**
30° 10′	.7012	2.1	.9368	.7	.7644	2.9	.2356	59° 50′
30° 20′	.7033	2.2	.9361	.8	.7673	2.8	.2327	59° 40′
30° 30′	.7055	2.1	.9353	.7	.7701	2.9	.2299	59° 30′
30° 40′	7076	2.1	.9346	.8	.7730	2.9	.2270	59° 20′
30° 50′	.7097	2.1	.9338	.7	.7759	2.9	.2241	59° 10′
31° 0′	9.7118	2.1	9.9331	.8	9.7788	2.8	0.2212	**59° 0′**
31° 10′	7139	2.1	.9323	.8	.7816	2.9	.2184	58° 50′
31° 20′	.7160	2.1	.9315	.7	.7845	2.8	.2155	58° 40′
31° 30′	.7181	2.0	.9308	.8	.7873	2.9	.2127	58° 30′
31° 40′	.7201	2.1	.9300	.8	.7902	2.8	.2098	58° 20′
31° 50′	.7222	2.0	.9292	.8	.7930	2.8	.2070	58° 10′
32° 0′	9.7242	2.0	9.9284	.8	9.7958	2.8	0.2042	**58° 0′**
32° 10′	.7262	2.0	.9276	.8	.7986	2.8	.2014	57° 50′
32° 20′	.7282	2.0	.9268	.8	.8014	2.8	.1986	57° 40′
32° 30′	.7302	2.0	.9260	.8	.8042	2.8	.1958	57° 30′
32° 40′	.7322	2.0	.9252	.8	.8070	2.7	.1930	57° 20′
32° 50′	.7342	1.9	.9244	.8	.8097	2.8	.1903	57° 10′
33° 0′	9.7361	1.9	9.9236	.8	9.8125	2.8	0.1875	**57° 0′**
33° 10′	.7380	2.0	.9228	.9	.8153	2.7	.1847	56° 50′
33° 20′	.7400	1.9	.9219	.8	.8180	2.8	.1820	56° 40′
33° 30′	.7419	1.9	.9211	.8	.8208	2.7	.1792	56° 30′
33° 40′	.7438	1.9	.9203	.9	.8235	2.8	.1765	56° 20′
33° 50′	.7457	1.9	.9194	.8	.8263	2.7	.1737	56° 10′
34° 0′	9.7476	1.8	9.9186	.9	9.8290	2.7	0.1710	**56° 0′**
34° 10′	.7494	1.9	.9177	.8	.8317	2.7	.1683	55° 50′
34° 20′	.7513	1.8	.9169	.9	.8344	2.7	.1656	55° 40′
34° 30′	.7531	1.9	.9160	.9	.8371	2.7	.1629	55° 30′
34° 40′	.7550	1.8	.9151	.9	.8398	2.7	.1602	55° 20′
34° 50′	.7568	1.8	.9142	.8	.8425	2.7	.1575	55° 10′
35° 0′	9.7586	1.8	9.9134	.9	9.8452	2.7	0.1548	**55° 0′**
35° 10′	.7604	1.8	.9125	.9	.8479	2.7	.1521	54° 50′
35° 20′	.7622	1.8	.9116	.9	.8506	2.7	.1494	54° 40′
35° 30′	.7640	1.7	.9107	.9	.8533	2.6	.1467	54° 30′
35° 40′	.7657	1.8	.9098	.9	.8559	2.7	.1441	54° 20′
35° 50′	.7675	1.7	.9089	.9	.8586	2.7	.1414	54° 10′
36° 0′	9.7692	1.8	9.9080	1.0	9.8613	2.6	0.1387	**54° 0′**
36° 10′	.7710	1.7	.9070	.9	.8639	2.7	.1361	53° 50′
36° 20′	.7727	1.7	.9061	.9	.8666	2.6	.1334	53° 40′
36° 30′	.7744	1.7	.9052	1.0	.8692	2.6	.1308	53° 30′
36° 40′	.7761	1.7	.9042	.9	.8718	2.7	.1282	53° 20′
36° 50′	.7778	1.7	.9033	1.0	.8745	2.6	.1255	53° 10′
37° 0′	9.7795	1.6	9.9023	.9	9.8771	2.6	0.1229	**53° 0′**
37° 10′	.7811	1.7	.9014	1.0	.8797	2.7	.1203	52° 50′
37° 20′	.7828	1.6	.9004	.9	.8824	2.6	.1176	52° 40′
37° 30′	.7844		.8995		.8850		.1150	52° 30′
	Cos.	D. 1′.	Sin.	D. 1′.	Cot.	D. 1′.	Tan.	Angle.

Digitized by Google

Angle.	Sin.	D. 1′.	Cos.	D. 1′.	Tan.	D. 1′.	Cot.	
37° 30′	9.7844	1.7	9.8995	1.0	9.8850	2.6	0.1150	52° 30′
37° 40′	.7861	1.6	.8985	1.0	.8876	2.6	.1124	52° 20′
37° 50′	.7877	1.6	.8975	1.0	.8902	2.6	.1098	52° 10′
38° 0′	9.7893	1.7	9.8965	1.0	9.8928	2.6	0.1072	**52° 0′**
38° 10′	.7910	1.6	.8955	1.0	.8954	2.6	.1046	51° 50′
38° 20′	.7926	1.5	.8945	1.0	.8980	2.6	.1020	51° 40′
38° 30′	.7941	1.6	.8935	1.0	.9006	2.6	.0994	51° 30′
38° 40′	.7957	1.6	.8925	1.0	.9032	2.6	.0968	51° 20′
38° 50′	.7973	1.6	.8915	1.0	.9058	2.6	.0942	51° 10′
39° 0′	9.7989	1.5	9.8905	1.0	9.9084	2.6	0.0916	**51° 0′**
39° 10′	.8004	1.6	.8895	1.1	.9110	2.5	.0890	50° 50′
39° 20′	.8020	1.5	.8884	1.0	.9135	2.6	.0865	50° 40′
39° 30′	.8035	1.5	.8874	1.0	.9161	2.6	.0839	50° 30′
39° 40′	.8050	1.6	.8864	1.1	.9187	2.5	.0813	50° 20′
39° 50′	.8066	1.5	.8853	1.0	.9212	2.6	.0788	50° 10′
40° 0′	9.8081	1.5	9.8843	1.1	9.9238	2.6	0.0762	**50° 0′**
40° 10′	.8096	1.5	.8832	1.1	.9264	2.5	.0736	49° 50′
40° 20′	.8111	1.4	.8821	1.1	.9289	2.6	.0711	49° 40′
40° 30′	.8125	1.5	.8810	1.0	.9315	2.6	.0685	49° 30′
40° 40′	.8140	1.5	.8800	1.1	.9341	2.5	.0659	49° 20′
40° 50′	.8155	1.4	.8789	1.1	.9366	2.6	.0634	49° 10′
41° 0′	9.8169	1.5	9.8778	1.1	9.9392	2.5	0.0608	**49° 0′**
41° 10′	.8184	1.4	.8767	1.1	.9417	2.6	.0583	48° 50′
41° 20′	.8198	1.5	.8756	1.1	.9443	2.5	.0557	48° 40′
41° 30′	.8213	1.4	.8745	1.2	.9468	2.6	.0532	48° 30′
41° 40′	.8227	1.4	.8733	1.1	.9494	2.5	.0506	48° 20′
41° 50′	.8241	1.4	.8722	1.1	.9519	2.5	.0481	48° 10′
42° 0′	9.8255	1.4	9.8711	1.2	9.9544	2.6	0.0456	**48° 0′**
42° 10′	.8269	1.4	.8699	1.1	.9570	2.5	.0430	47° 50′
42° 20′	.8283	1.4	.8688	1.2	.9595	2.6	.0405	47° 40′
42° 30′	.8297	1.4	.8676	1.1	.9621	2.5	.0379	47° 30′
42° 40′	.8311	1.3	.8665	1.2	.9646	2.5	.0354	47° 20′
42° 50′	.8324	1.4	.8653	1.2	.9671	2.6	.0329	47° 10′
43° 0′	9.8338	1.3	9.8641	1.2	9.9697	2.5	0.0303	**47° 0′**
43° 10′	.8351	1.4	.8629	1.1	.9722	2.5	.0278	46° 50′
43° 20′	.8365	1.3	.8618	1.2	.9747	2.5	.0253	46° 40′
43° 30′	.8378	1.3	.8606	1.2	.9772	2.6	.0228	46° 30′
43° 40′	.8391	1.4	.8594	1.2	.9798	2.5	.0202	46° 20′
43° 50′	.8405	1.3	.8582	1.3	.9823	2.5	.0177	46° 10′
44° 0′	9.8418	1.3	9.8569	1.2	9.9848	2.6	0.0152	**46° 0′**
44° 10′	.8431	1.3	.8557	1.2	.9874	2.5	.0126	45° 50′
44° 20′	.8444	1.3	.8545	1.3	.9899	2.5	.0101	45° 40′
44° 30′	.8457	1.2	.8532	1.2	.9924	2.5	.0076	45° 30′
44° 40′	.8469	1.3	.8520	1.3	.9949	2.6	.0051	45° 20′
44° 50′	.8482	1.3	.8507	1.2	.9975	2.5	.0025	45° 10′
45° 0′	9.8495		9.8495		0.0000		0.0000	**45° 0′**
	Cos.	**D. 1′.**	**Sin.**	**D. 1′.**	**Cot.**	**D. 1′.**	**Tan.**	**Angle.**

Digitized by Google

A.	Sin.	Cos.		A.	Sin.	Cos.		A.	Sin.	Cos.	
0°	.000000	1.0000	90°	30′	.1305	.9914	30′	15°	.2588	.9659	75°
10′	.002909	1.0000	50′	40′	.1334	.9911	20′	10′	.2616	.9652	50′
20′	.005818	1.0000	40′	50′	.1363	.9907	10′	20′	.2644	.9644	40′
30′	.008727	1.0000	30′	8°	.1392	.9903	82°	30′	.2672	.9636	30′
40′	.011635	.9999	20′	10′	.1421	.9899	50′	40′	.2700	.9628	20′
50′	.014544	.9999	10′	20′	.1449	.9894	40′	50′	.2728	.9621	10′
1°	.017452	.9998	89°	30′	.1478	.9890	30′	16°	.2756	.9613	74°
10′	.02036	.9998	50′	40′	.1507	.9886	20′	10′	.2784	.9605	50′
20′	.02327	.9997	40′	50′	.1536	.9881	10′	20′	.2812	.9596	40′
30′	.02618	.9997	30′	9°	.1564	.9877	81°	30′	.2840	.9588	30′
40′	.02908	.9996	20′	10′	.1593	.9872	50′	40′	.2868	.9580	20′
50′	.03199	.9995	10′	20′	.1622	.9868	40′	50′	.2896	.9572	10′
2°	.03490	.9994	88°	30′	.1650	.9863	30′	17°	.2924	.9563	73°
10′	.03781	.9993	50′	40′	.1679	.9858	20′	10′	.2952	.9555	50′
20′	.04071	.9992	40′	50′	.1708	.9853	10′	20′	.2979	.9546	40′
30′	.04362	.9990	30′	10°	.1736	.9848	80°	30′	.3007	.9537	30′
40′	.04653	.9989	20′	10′	.1765	.9843	50′	40′	.3035	.9528	20′
50′	.04943	.9988	10′	20′	.1794	.9838	40′	50′	.3062	.9520	10′
3°	.05234	.9986	87°	30′	.1822	.9833	30′	18°	.3090	.9511	72°
10′	.05524	.9985	50′	40′	.1851	.9827	20′	10′	.3118	.9502	50′
20′	.05814	.9983	40′	50′	.1880	.9822	10′	20′	.3145	.9492	40′
30′	.06105	.9981	30′	11°	.1908	.9816	79°	30′	.3173	.9483	30′
40′	.06395	.9980	20′	10′	.1937	.9811	50′	40′	.3201	.9474	20′
50′	.06685	.9978	10′	20′	.1965	.9805	40′	50′	.3228	.9465	10′
4°	.06976	.9976	86°	30′	.1994	.9799	30′	19°	.3256	.9455	71°
10′	.07266	.9974	50′	40′	.2022	.9793	20′	10′	.3283	.9446	50′
20′	.07556	.9971	40′	50′	.2051	.9787	10′	20′	.3311	.9436	40′
30′	.07846	.9969	30′	12°	.2079	.9781	78°	30′	.3338	.9426	30′
40′	.08136	.9967	20′	10′	.2108	.9775	50′	40′	.3365	.9417	20′
50′	.08426	.9964	10′	20′	.2136	.9769	40′	50′	.3393	.9407	10′
5°	.08716	.9962	85°	30′	.2164	.9763	30′	20°	.3420	.9397	70°
10′	.09005	.9959	50′	40′	.2193	.9757	20′	10′	.3448	.9387	50′
20′	.09295	.9957	40′	50′	.2221	.9750	10′	20′	.3475	.9377	40′
30′	.09585	.9954	30′	13°	.2250	.9744	77°	30′	.3502	.9367	30′
40′	.09874	.9951	20′	10′	.2278	.9737	50′	40′	.3529	.9356	20′
50′	.10164	.9948	10′	20′	.2306	.9730	40′	50′	.3557	.9346	10′
6°	.10453	.9945	84°	30′	.2334	.9724	30′	21°	.3584	.9336	69°
10′	.10742	.9942	50′	40′	.2363	.9717	20′	10′	.3611	.9325	50′
20′	.11031	.9939	40′	50′	.2391	.9710	10′	20′	.3638	.9315	40′
30′	.11320	.9936	30′	14°	.2419	.9703	76°	30′	.3665	.9304	30′
40′	.11609	.9932	20′	10′	.2447	.9696	50′	40′	.3692	.9293	20′
50′	.11898	.9929	10′	20′	.2476	.9689	40′	50′	.3719	.9283	10′
7°	.12187	.9925	83°	30′	.2504	.9681	30′	22°	.3746	.9272	68°
10′	.12476	.9922	50′	40′	.2532	.9674	20′	10′	.3773	.9261	50′
20′	.12764	.9918	40′	50′	.2560	.9667	10′	20′	.3800	.9250	40′
30′	.13053	.9914	30′	15°	.2588	.9659	75°	30′	.3827	.9239	30′
	Cos.	Sin.	A.		Cos.	Sin.	A.		Cos.	Sin.	A.

Digitized by Google

A.	Sin.	Cos.		A.	Sin.	Cos.		A.	Sin.	Cos.	
30′	.3827	.9239	30′	**30°**	.5000	.8660	**60°**	30′	.6088	.7934	30′
40′	.3854	.9228	20′	10′	.5025	.8646	50′	40′	.6111	.7916	20′
50′	.3881	.9216	10′	20′	.5050	.8631	40′	50′	.6134	.7898	10′
23°	.3907	.9205	**67°**	30′	.5075	.8616	30′	**38°**	.6157	.7880	**52°**
10′	.3934	.9194	50′	40′	.5100	.8601	20′	10′	.6180	.7862	50′
20′	.3961	.9182	40′	50′	.5125	.8587	10′	20′	.6202	.7844	40′
30′	.3987	.9171	30′	**31°**	.5150	.8572	**59°**	30′	.6225	.7826	30′
40′	.4014	.9159	20′	10′	.5175	.8557	50′	40′	.6248	.7808	20′
50′	.4041	.9147	10′	20′	.5200	.8542	40′	50′	.6271	.7790	10′
24°	.4067	.9135	**66°**	30′	.5225	.8526	30′	**39°**	.6293	.7771	**51°**
10′	.4094	.9124	50′	40′	.5250	.8511	20′	10′	.6316	.7753	50′
20′	.4120	.9112	40′	50′	.5275	.8496	10′	20′	.6338	.7735	40′
30′	.4147	.9100	30′	**32°**	.5299	.8480	**58°**	30′	.6361	.7716	30′
40′	.4173	.9088	20′	10′	.5324	.8465	50′	40′	.6383	.7698	20′
50′	.4200	.9075	10′	20′	.5348	.8450	40′	50′	.6406	.7679	10′
25°	.4226	.9063	**65°**	30′	.5373	.8434	30′	**40°**	.6428	.7660	**50°**
10′	.4253	.9051	50′	40′	.5398	.8418	20′	10′	.6450	.7642	50′
20′	.4279	.9038	40′	50′	.5422	.8403	10′	20′	.6472	.7623	40′
30′	.4305	.9026	30′	**33°**	.5446	.8387	**57°**	30′	.6494	.7604	30′
40′	.4331	.9013	20′	10′	.5471	.8371	50′	40′	.6517	.7585	20′
50′	.4358	.9001	10′	20′	.5495	.8355	40′	50′	.6539	.7566	10′
26°	.4384	.8988	**64°**	30′	.5519	.8339	30′	**41°**	.6561	.7547	**49°**
10′	.4410	.8975	50′	40′	.5544	.8323	20′	10′	.6583	.7528	50′
20′	.4436	.8962	40′	50′	.5568	.8307	10′	20′	.6604	.7509	40′
30′	.4462	.8949	30′	**34°**	.5592	.8290	**56°**	30′	.6626	.7490	30′
40′	.4488	.8936	20′	10′	.5616	.8274	50′	40′	.6648	.7470	20′
50′	.4514	.8923	10′	20′	.5640	.8258	40′	50′	.6670	.7451	10′
27°	.4540	.8910	**63°**	30′	.5664	.8241	30′	**42°**	.6691	.7431	**48°**
10′	.4566	.8897	50′	40′	.5688	.8225	20′	10′	.6713	.7412	50′
20′	.4592	.8884	40′	50′	.5712	.8208	10′	20′	.6734	.7392	40′
30′	.4617	.8870	30′	**35°**	.5736	.8192	**55°**	30′	.6756	.7373	30′
40′	.4643	.8857	20′	10′	.5760	.8175	50′	40′	.6777	.7353	20′
50′	.4669	.8843	10′	20′	.5783	.8158	40′	50′	.6799	.7333	10′
28°	.4695	.8829	**62°**	30′	.5807	.8141	30′	**43°**	.6820	.7314	**47°**
10′	.4720	.8816	50′	40′	.5831	.8124	20′	10′	.6841	.7294	50′
20′	.4746	.8802	40′	50′	.5854	.8107	10′	20′	.6862	.7274	40′
30′	.4772	.8788	30′	**36°**	.5878	.8090	**54°**	30′	.6884	.7254	30′
40′	.4797	.8774	20′	10′	.5901	.8073	50′	40′	.6905	.7234	20′
50′	.4823	.8760	10′	20′	.5925	.8056	40′	50′	.6926	.7214	10′
29°	.4848	.8746	**61°**	30′	.5948	.8039	30′	**44°**	.6947	.7193	**46°**
10′	.4874	.8732	50′	40′	.5972	.8021	20′	10′	.6967	.7173	50′
20′	.4899	.8718	40′	50′	.5995	.8004	10′	20′	.6988	.7153	40′
30′	.4924	.8704	30′	**37°**	.6018	.7986	**53°**	30′	.7009	.7133	30′
40′	.4950	.8689	20′	10′	.6041	.7969	50′	40′	.7030	.7112	20′
50′	.4975	.8675	10′	20′	.6065	.7951	40′	50′	.7050	.7092	10′
30°	.5000	.8660	**60°**	30′	.6088	.7934	30′	**45°**	.7071	.7071	**45°**
	Cos.	Sin.	A.		Cos.	Sin.	A.		Cos.	Sin.	A.

Digitized by Google

A.	Tan.	Cot.		A.	Tan.	Cot.		A.	Tan.	Cot.	
0°	.000000	∞	90°	30′	.1317	7.5958	30′	15°	.2679	3.7321	75°
10′	.002909	343.7737	50′	40′	.1346	7.4287	20′	10′	.2711	3.6891	50′
20′	.005818	171.8854	40′	50′	.1376	7.2687	10′	20′	.2742	3.6470	40′
30′	.008727	114.5887	30′	8°	.1405	7.1154	82°	30′	.2773	3.6059	30′
40′	.011636	85.9398	20′	10′	.1435	6.9682	50′	40′	.2805	3.5656	20′
50′	.014545	68.7501	10′	20′	.1465	6.8269	40′	50′	.2836	3.5261	10′
1°	.017455	57.2900	89°	30′	.1495	6.6912	30′	16°	.2867	3.4874	74°
10′	.02036	49.1039	50′	40′	.1524	6.5606	20′	10′	.2899	3.4495	50′
20′	.02328	42.9641	40′	50′	.1554	6.4348	10′	20′	.2931	3.4124	40′
30′	.02619	38.1885	30′	9°	.1584	6.3138	81°	30′	.2962	3.3759	30′
40′	.02910	34.3678	20′	10′	.1614	6.1970	50′	40′	.2994	3.3402	20′
50′	.03201	31.2416	10′	20′	.1644	6.0844	40′	50′	.3026	3.3052	10′
2°	.03492	28.6363	88°	30′	.1673	5.9758	30′	17°	.3057	3.2709	73°
10′	.03783	26.4316	50′	40′	.1703	5.8708	20′	10′	.3089	3.2371	50′
20′	.04075	24.5418	40′	50′	.1733	5.7694	10′	20′	.3121	3.2041	40′
30′	.04366	22.9038	30′	10°	.1763	5.6713	80°	30′	.3153	3.1716	30′
40′	.04658	21.4704	20′	10′	.1793	5.5764	50′	40′	.3185	3.1397	20′
50′	.04949	20.2056	10′	20′	.1823	5.4845	40′	50′	.3217	3.1084	10′
3°	.05241	19.0811	87°	30′	.1853	5.3955	30′	18°	.3249	3.0777	72°
10′	.05533	18.0750	50′	40′	.1883	5.3093	20′	10′	.3281	3.0475	50′
20′	.05824	17.1693	40′	50′	.1914	5.2257	10′	20′	.3314	3.0178	40′
30′	.06116	16.3499	30′	11°	.1944	5.1446	79°	30′	.3346	2.9887	30′
40′	.06408	15.6048	20′	10′	.1974	5.0658	50′	40′	.3378	2.9600	20′
50′	.06700	14.9244	10′	20′	.2004	4.9894	40′	50′	.3411	2.9319	10′
4°	.06993	14.3007	86°	30′	.2035	4.9152	30′	19°	.3443	2.9042	71°
10′	.07285	13.7267	50′	40′	.2065	4.8430	20′	10′	.3476	2.8770	50′
20′	.07578	13.1969	40′	50′	.2095	4.7729	10′	20′	.3508	2.8502	40′
30′	.07870	12.7062	30′	12°	.2126	4.7046	78°	30′	.3541	2.8239	30′
40′	.08163	12.2505	20′	10′	.2156	4.6382	50′	40′	.3574	2.7980	20′
50′	.08456	11.8262	10′	20′	.2186	4.5736	40′	50′	.3607	2.7725	10′
5°	.08749	11.4301	85°	30′	.2217	4.5107	30′	20°	.3640	2.7475	70°
10′	.09042	11.0594	50′	40′	.2247	4.4494	20′	10′	.3673	2.7228	50′
20′	.09335	10.7119	40′	50′	.2278	4.3897	10′	20′	.3706	2.6985	40′
30′	.09629	10.3854	30′	13°	.2309	4.3315	77°	30′	.3739	2.6746	30′
40′	.09923	10.0780	20′	10′	.2339	4.2747	50′	40′	.3772	2.6511	20′
50′	.10216	9.7882	10′	20′	.2370	4.2193	40′	50′	.3805	2.6279	10′
6°	.10510	9.5144	84°	30′	.2401	4.1653	30′	21°	.3839	2.6051	69°
10′	.10805	9.2553	50′	40′	.2432	4.1126	20′	10′	.3872	2.5826	50′
20′	.11099	9.0098	40′	50′	.2462	4.0611	10′	20′	.3906	2.5605	40′
30′	.11394	8.7769	30′	14°	.2493	4.0108	76°	30′	.3939	2.5386	30′
40′	.11688	8.5555	20′	10′	.2524	3.9617	50′	40′	.3973	2.5172	20′
50′	.11983	8.3450	10′	20′	.2555	3.9136	40′	50′	.4006	2.4960	10′
7°	.12278	8.1443	83°	30′	.2586	3.8667	30′	22°	.4040	2.4751	68°
10′	.12574	7.9530	50′	40′	.2617	3.8208	20′	10′	.4074	2.4545	50′
20′	.12869	7.7704	40′	50′	.2648	3.7760	10′	20′	.4108	2.4342	40′
30′	.13165	7.5958	30′	15°	.2679	3.7321	75°	30′	.4142	2.4142	30′
	Cot.	Tan.	A.		Cot.	Tan.	A.		Cot.	Tan.	A.

Digitized by Google

A.	Tan.	Cot.		A.	Tan.	Cot.		A.	Tan.	Cot.	
30′	.4142	2.4142	30′	**30°**	.5774	1.7321	**60°**	30′	.7673	1.3032	30′
40′	.4176	2.3945	20′	10′	.5812	1.7205	50′	40′	.7720	1.2954	20′
50′	.4210	2.3750	10′	20′	.5851	1.7090	40′	50′	.7766	1.2876	10′
23°	.4245	2.3559	**67°**	30′	.5890	1.6977	30′	**38°**	.7813	1.2799	**52°**
10′	.4279	2.3369	50′	40′	.5930	1.6864	20′	10′	.7860	1.2723	50′
20′	.4314	2.3183	40′	50′	.5969	1.6753	10′	20′	.7907	1.2647	40′
30′	.4348	2.2998	30′	**31°**	.6009	1.6643	**59°**	30′	.7954	1.2572	30′
40′	.4383	2.2817	20′	10′	.6048	1.6534	50′	40′	.8002	1.2497	20′
50′	.4417	2.2637	10′	20′	.6088	1.6426	40′	50′	.8050	1.2423	10′
24°	.4452	2.2460	**66°**	30′	.6128	1.6319	30′	**39°**	.8098	1.2349	**51°**
10′	.4487	2.2286	50′	40′	.6168	1.6212	20′	10′	.8146	1.2276	50′
20′	.4522	2.2113	40′	50′	.6208	1.6107	10′	20′	.8195	1.2203	40′
30′	.4557	2.1943	30′	**32°**	.6249	1.6003	**58°**	30′	.8243	1.2131	30′
40′	.4592	2.1775	20′	10′	.6289	1.5900	50′	40′	.8292	1.2059	20′
50′	.4628	2.1609	10′	20′	.6330	1.5798	40′	50′	.8342	1.1988	10′
25°	.4663	2.1445	**65°**	30′	.6371	1.5697	30′	**40°**	.8391	1.1918	**50°**
10′	.4699	2.1283	50′	40′	.6412	1.5597	20′	10′	.8441	1.1847	50′
20′	.4734	2.1123	40′	50′	.6453	1.5497	10′	20′	.8491	1.1778	40′
30′	.4770	2.0965	30′	**33°**	.6494	1.5399	**57°**	30′	.8541	1.1708	30′
40′	.4806	2.0809	20′	10′	.6536	1.5301	50′	40′	.8591	1.1640	20′
50′	.4841	2.0655	10′	20′	.6577	1.5204	40′	50′	.8642	1.1571	10′
26°	.4877	2.0503	**64°**	30′	.6619	1.5108	30′	**41°**	.8693	1.1504	**49°**
10′	.4913	2.0353	50′	40′	.6661	1.5013	20′	10′	.8744	1.1436	50′
20′	.4950	2.0204	40′	50′	.6703	1.4919	10′	20′	.8796	1.1369	40′
30′	.4986	2.0057	30′	**34°**	.6745	1.4826	**56°**	30′	.8847	1.1303	30′
40′	.5022	1.9912	20′	10′	.6787	1.4733	50′	40′	.8899	1.1237	20′
50′	.5059	1.9768	10′	20′	.6830	1.4641	40′	50′	.8952	1.1171	10′
27°	.5095	1.9626	**63°**	30′	.6873	1.4550	30′	**42°**	.9004	1.1106	**48°**
10′	.5132	1.9486	50′	40′	.6916	1.4460	20′	10′	.9057	1.1041	50′
20′	.5169	1.9347	40′	50′	.6959	1.4370	10′	20′	.9110	1.0977	40′
30′	.5206	1.9210	30′	**35°**	.7002	1.4281	**55°**	30′	.9163	1.0913	30′
40′	.5243	1.9074	20′	10′	.7046	1.4193	50′	40′	.9217	1.0850	20′
50′	.5280	1.8940	10′	20′	.7089	1.4106	40′	50′	.9271	1.0786	10′
28°	.5317	1.8807	**62°**	30′	.7133	1.4019	30′	**43°**	.9325	1.0724	**47°**
10′	.5354	1.8676	50′	40′	.7177	1.3934	20′	10′	.9380	1.0661	50′
20′	.5392	1.8546	40′	50′	.7221	1.3848	10′	20′	.9435	1.0599	40′
30′	.5430	1.8418	30′	**36°**	.7265	1.3764	**54°**	30′	.9490	1.0538	30′
40′	.5467	1.8291	20′	10′	.7310	1.3680	50′	40′	.9545	1.0477	20′
50′	.5505	1.8165	10′	20′	.7355	1.3597	40′	50′	.9661	1.0416	10′
29°	.5543	1.8040	**61°**	30′	.7400	1.3514	30′	**44°**	.9657	1.0355	**46°**
10′	.5581	1.7917	50′	40′	.7445	1.3432	20′	10′	.9713	1.0295	50′
20′	.5619	1.7796	40′	50′	.7490	1.3351	10′	20′	.9770	1.0235	40′
30′	.5658	1.7675	30′	**37°**	.7536	1.3270	**53°**	30′	.9827	1.0176	30′
40′	.5696	1.7556	20′	10′	.7581	1.3190	50′	40′	.9884	1.0117	20′
50′	.5735	1.7437	10′	20′	.7627	1.3111	40′	50′	.9942	1.0058	10′
30°	.5774	1.7321	**60°**	30′	.7673	1.3032	30′	**45°**	1.0000	1.0000	**45°**
	Cot.	Tan.	A.		Cot.	Tan.	A.		Cot.	Tan.	A.

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

THIS BOOK IS DUE ON THE LAST DATE STAMPED BELOW

LD INITIAL FINE OF 25 CENTS
WILL BE ASSESSED FOR FAILURE TO RETURN THIS BOOK ON THE DATE DUE. THE PENALTY WILL INCREASE TO 50 CENTS ON THE FOURTH DAY AND TO $1.00 ON THE SEVENTH DAY OVERDUE.

MAY 31 1947
21Dec'48WM
4May'49WW
8Nov'49LS
7NovDEAD
24Aug'53VL
AUG 26 1953 LU
19Apr'55JP
APR 5 1955 LU
APR 5 1955 LU
24Jan'56LT
JAN 27 1956 LU
31Aug'56MR
AUG 7 1956 LU
22Jul'57MR
REC'D LD
JUL 8 1957

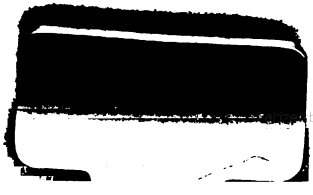
REC'D LD
JUL 8 1957
30Oc'59ES
IN STACKS
OCT 16 1959
REC'D LD
FEB 8 1960
REC'D LD
OCT 1 1962

LD 21-100m-12,'43 (8796s)

Digitized by Google

YB 17148

THE UNIVERSITY OF CALIFORNIA LIBRARY



Digitized by Google